A note on pricing forwards in a regime-switching model via integro-PDE method

Fred Espen Benth*[2], Rüdiger Kiesel†[1], Anna Nazarova‡[1]

[2]Centre of Mathematics for Applications, University of Oslo, PO Box 1053 Blindern, N-0316 Oslo, Norway.

[1]Chair of Energy Trading and Financial Services, University Duisburg-Essen, Universitätstrasse 12, 45141 Essen, Germany.

January 26, 2014

Abstract.

Keywords: Power forwards, regime switching, integro-pde, energy markets

1. Motivation and introduction

In Geman and Roncoroni (2006) a Markov model for the spot price dynamics of electricity is proposed. This model is a standard Ornstein-Uhlenbeck process, with a non-standard jump term. The jumps are designed to model the spikes frequently observed for electricity spot prices.

This threshold model is a Markov jump-diffusion model, however, not feasible for explicit pricing of forward contracts due to its specification. A forward contract is an agreement where the buyer purchases any specified commodity at an agreed time to an agreed price. The agreed price, commonly known as the *forward price* is the price such that the current value of the contract is zero. With a pricing measure it can be viewed as the best predicted spot time at time of the transaction. Mathematically it is expressed as the conditional expected spot price (possibly under a risk-neutral probability).

The problem we focus here is the derivation of the forward price dynamics. Unlike many other models, the threshold model does not allow for explicit calculation of the forward price, and numerical methods are called for. Since we want to find the dynamics, the Monte Carlo method is very cumbersome, and we are going to analyse a PDE-based approach. Since the threshold model involves jumps, we are led to integro-PDEs, and numerical methods for such. We want to compare the resulting forward prices with those of similar models.

*E-mail: fredb@math.uio.no

†E-mail: ruediger.kiesel@uni-due.de

‡E-mail: anna.nazarova@uni-due.de

2. Mathematical formulation

2.1. **Technical preliminaries.** In this section we recall some definitions and helpful facts we use in further sections. We mainly consider a Lévy-type process for modelling the power price. Let $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\})$ be a complete filtered probability space. Let T defines the time horizon. We will use the following definitions given in Chapter 3 in Cont and Tankov (2004).

Definition 1. *A cádlág stochastic process $(L_t)_{t\geq 0}$ on $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\})$ with values in $\mathbb{R}$ such that $L_0 = 0$ is called a Lévy process if it satisfies the following properties:*

(1) Independent increments: for every increasing sequence of times $t_0, \dots, t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent;

(2) Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t;

(3) Stochastic continuity: $\forall \epsilon > 0,\ \lim_{h\to 0} \mathbb{P}\left(|X_{t+h} - X_t| \geq 0\right) = 0$.

A Lévy process is associated with its Lévy measure ν

Definition 2. *Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The measure ν on $\mathbb{R}$ defined by:*

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1] : \Delta L_t \neq 0, \quad \Delta L_t \in A\}\right], \quad A \in \mathcal{B}(\mathbb{R}) \tag{1}$$

is called the Lévy measure of L: $\nu(A)$ is the expected number, per unit time, of jumps whose size belongs to A.

Every Lévy process is characterised by its characteristic triplet (γ, b, ν), where $\gamma \in \mathbb{R}$ is the drift term, $b \in \mathbb{R}_{\geq 0}$ is the diffusion coefficient and ν is the Lévy measure. Next useful theorem is a celebrated Lévy-Khincin decomposition.

Theorem 1. *Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with a characteristic triplet (γ, b, ν). Then*

$$\mathbb{E}\left[e^{izL_t}\right] = e^{t\,\psi(z)}, \quad z \in \mathbb{R}, \tag{2}$$

with

$$\psi(z) = i\gamma z - \frac{z^2 b}{2} + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{|x|<1}\right) \nu(\,\mathrm{d}x). \tag{3}$$

Proof. The proof can be found in various sources, for example in Chapter 3 in Cont and Tankov (2004) and in Chapter 4 in Sato (1999). □

According to this theorem a Lévy process can always be decomposed into several processes: a deterministic linear process (drift) with parameter γ, a Brownian motion with coefficient $\sqrt{b}$, a compound Poisson process with arrival rate $\lambda := \nu\left(\mathbb{R}\backslash(-1,1)\right)$ and jump size distribution given by its cumulative distribution function $\mathbb{F}(\,\mathrm{d}x) := \frac{\nu(\,\mathrm{d}x)}{\nu(\mathbb{R}\backslash(-1,1))}\,\mathbf{1}_{|x|\geq 1}$, and the last component: pure jump martingale process.

Now let us recall a few facts about the Lévy measure ν:

- the Lévy measure ν on $\mathbb{R}$ satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \left(1 \wedge (|x|)^2\right) \nu(\,\mathrm{d}x) < \infty$.

- if ν is a finite measure, i.e. $\lambda = \nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(\mathrm{d}x) < \infty$, then $\mathbb{F}(\mathrm{d}x) := \frac{\nu(\mathrm{d}x)}{\lambda}$ is a probability measure. Then λ is interpreted as the expected number of jumps and $\mathbb{F}(\mathrm{d}x)$ is the distribution of the jump size x. It is also said the the Lévy process L_t has finite activity. (Theorem 21.3 in Sato (1999)).
- if $b \neq 0$ or $\int_{|x|\leq 1} |x|\ \nu(\mathrm{d}x) = \infty$, then almost all the paths of the Lévy process L_t have infinite variation. (Theorem 21.9 in Sato (1999)).

Proposition 2.1. *Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$ with a characteristic triplet (γ, b, ν). Then*

(1) $\mathbb{E}[|L_t|^p] < \infty$ if and only if $\int_{|x|\geq 1} |x|^p\ \nu(\mathrm{d}x) < \infty$;

(2) $\mathbb{E}[e^{pL_t}] < \infty$ if and only if $\int_{|x|\geq 1} e^{px}\ \nu(\mathrm{d}x) < \infty$.

Proof. Theorem 25.3 in Sato (1999). □

We now continue with a few more known facts and properties of a Lévy process and its connection to the martingale theory.

- a *semimartingale* is a stochastic process $(X_t)_{0\leq t\leq T}$ that can be represented as
$$X = X_0 + M + A,$$
where X_0 is finite and $\mathcal{F}$-measurable, M is a local martingale with $M_0 = 0$ and A is a finite variation process with $A_0 = 0$;
- a semimartingale X is a *special semimartingale*, if the process A is predictable;
- every Lévy process is a semimartingale due to its Lévy-Khincin decomposition;
- every Lévy process with its finite first moment (i.e. if and only if $\int_{|x|\geq 1} |x|\ \nu(\mathrm{d}x) < \infty$;) is also a special semimartingale;
- the following three assertions are equivalent:
  (1) a Lévy process L_t is a special semimartingale;
  (2) $\int_{\mathbb{R}} \left(|x| \wedge |x|^2\right)\ \nu(\mathrm{d}x) < \infty$,
  (3) $\int_{\mathbb{R}} \left(|x|\, \mathbf{1}_{|x|\geq 1}\right)\ \nu(\mathrm{d}x) < \infty$,
  this is a consequence of Lemma 2.8 in Kallsen and Shiryaev (2002).

Theorem 2. *Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. Define*
$$L_t^* = \sup_{s\in[0,t]} |L_s|\,.$$
Let $g(r)$ be a nonnegative continuous submultiplicative function on $[0, \infty)$, increasing to ∞ as $r \to \infty$. Then the following four statements are equivalent:

(1) $\mathbb{E}[g(L_t^*)] < \infty$ for some $t > 0$;

(2) $\mathbb{E}[g(L_t^*)] < \infty$ for every $t > 0$;

(3) $\mathbb{E}[g(|L_t^*)|] < \infty$ for some $t > 0$;

(4) $\mathbb{E}[g(|L_t^*|)] < \infty$ *for every* $t > 0$.

Proof. Theorem 25.18 in Sato (1999). □

2.2. **Power price modelling.** Let $S(t)$ be the spot price of electricity (power) defined as

$$S(t) = \exp(\mu(t) + Y(t)), \tag{4}$$

where $\mu(t)$ is some deterministic seasonality function and $Y(t)$ is some stochastic process. The classical approach to model the process $Y(t)$, as for example considered in Cartea and Figueroa (2005), is stated as

$$dY(t) = -\alpha Y(t)\ \mathrm{d}t + \sigma\ \mathrm{d}W_t +\ \mathrm{d}Q_t, \tag{5}$$

where W_t is a Brownian motion, Q_t is a compound Poisson process, α is the speed of mean-reversion and σ is the volatility. Process W_t and Q_t are assumed to be mutually independent.

Alternatively, Geman and Roncoroni (2006) proposed to model this stochastic component differently

$$dX(t) = -\alpha X(t)\ \mathrm{d}t + \sigma\ \mathrm{d}W_t + h(X(t))\ \mathrm{d}Q_t, \tag{6}$$

where all the notations and assumptions are the same as for the process $Y(t)$, and $h(x)$ is a state-dependent function which is -1 for large values of X (defined by some threshold $\mathcal{T}$) and 1 otherwise. Despite the "regime-switching" term $h(x)$, this process holds the Markov property in a single state variable, for a proof see Roncoroni (2002). The authors claim that the process X_t is a special semimartingale. This model is referred here as the threshold model.

The difference of the two models lies in the change of sign of the h-function. This function ensures that the price may jump downwards in the case of high spot prices. Note that when we use a minus in front of the jumps in the classical model, we want to use it in the "high price" regime for the threshold model. By the "high price" regime we mean here that the price is far above its mean level, which can happen when a spike or big jump occurred.

We also notice here that the process $L_t := \sigma W_t + Q_t$ is a Lévy process in a contrast to the process $L_t^h := \sigma W_t + h(x)\, Q_t$ which does not satisfy the properties given in Definition 1.

2.3. **Forward modelling.** We know that the forward price $F(t,T)$ at time t, for a contract with a delivery at time $T \geq t$, is

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S(T) \,|\, \mathcal{F}_t], \tag{7}$$

which is a martingale under the equivalent martingale measure $\mathbb{Q}$. However, the power spot price does not need to be a martinagale with respect to $\mathbb{Q}$ since power is a non-storable commodity. So we call the measure $\mathbb{Q}$ a pricing measure, as it is a probability that takes into account all the risks associated with the change in the price (spikes can, for example, happen due to sudden weather change or unexpected outage of equipment). The choice of $\mathbb{Q}$ can be done via a canonical Girsanov (drift part) and Esscher (jump part) transformations. Alternatively, one could say that the process S_t is already under the measure $\mathbb{Q}$ and one could argue that the market will charge an additional risk premium by changing/adjusting the mean level. This would mean the $\mathbb{Q} = \mathbb{P}$ with the latter being the real-world pricing measure. From now on we apply this assumption.

By the Markovian property we can write the forward price explicitly as a function of $X(t)$ as

$$F(t,T,X(t)) = e^{\mu(T)}\,\mathbb{E}\,[e^{X(T)} \,|\, X(t)] = e^{\mu(T)}\,f(t,X(t)),$$

with $f(t,X(t)) := \mathbb{E}\,[e^{X(T)} \,|\, X(t) = x]$. Our aim here is to derive efficient routines to calculate the function $f(t,X(t))$ based on the associated integro-PDE in terms of the threshold model and to study the impact of the function $h(x)$ on the forward prices. We will also compare obtained forwards with ones from the standard classical model given in Equation (5).

2.4. **Forward price of the classical model.** This model allows for explicit forward price formula. More precisely, let the forward price be

$$G(t,T,Y(t)) = e^{\mu(T)}\,\mathbb{E}\,[e^{Y(T)} \,|\, Y(t)] = e^{\mu(T)}\,g(t,Y(t)), \tag{8}$$

with $g(t,Y(t)) \equiv \mathbb{E}\,[e^{Y(T)} \,|\, Y(t) = y]$. One can calculate the function $g(t,Y(t))$ analytically by appealing to the moment generating function of the compound Poisson process Q_t. We start with the dynamics of the logarithm of the price S_t

$$\begin{aligned} \mathrm{dlog} S_t &= \mu_t'\,\mathrm{d}t - \alpha\big(\log S_t - \mu(t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}Q_t \\ &= \alpha\big(\hat{\mu}(t) - \log S_t\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + \mathrm{d}Q_t, \end{aligned} \tag{9}$$

where $\hat{\mu}(t) := \frac{1}{\alpha}\mu_t' + \mu(t)$. Let us now apply Ito's lemma to $(e^{\alpha t}\log S_t)$ to obtain

$$\mathrm{d}(e^{\alpha t}\log S_t) = \alpha e^{\alpha t}\hat{\mu}(t)\,\mathrm{d}t + e^{\alpha t}\sigma\,\mathrm{d}W_t + e^{\alpha t}\,\mathrm{d}Q_t. \tag{10}$$

After integrating from t to T and replacing terms, we have

$$\log S_T = \mu(T) + Y(t)e^{-\alpha(T-t)} + \sigma \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}Q_s, \tag{11}$$

then the price S_T becomes

$$S_T = e^{\mu(T)+Y(T)} = e^{\mu(T)+Y(t)e^{-\alpha(T-t)}+\sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s+\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}Q_s}. \tag{12}$$

Now coming back to Equation (8) we have that function $g(t, Y(t))$ becomes

$$\begin{aligned} g(t,Y(t)) &\equiv \mathbb{E}\left[\exp(Y(T))\,|\,Y(t)=y\right] \\ &= \mathbb{E}\left[e^{Y(t)e^{-\alpha(T-t)}}\, e^{\sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s}\, e^{\int_t^T Je^{-\alpha(T-s)}\,\mathrm{d}N_s}\,|\,Y(t)=y\right] \\ &= e^{ye^{-\alpha(T-t)}} \exp\Big(\frac{\sigma^2}{4\alpha}\big(1-e^{-2\alpha(T-t)}\big)\Big) \exp\Big(\lambda\int_t^T (\mathbb{E}[e^{Je^{-\alpha(T-s)}}]-1)\,\mathrm{d}s\Big), \end{aligned} \tag{13}$$

where for the last equality we use mutual independence of W_t and Q_t, the fact the $\sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s$ is a normally distributed random variable and the Lévy-Khincin representation for the compound Poisson process Q_t.

2.4.1. *Normal distribution for jumps.* Now following Cartea and Figueroa (2005), we assume Normal distribution for the jump size J, i.e. $J \sim \mathcal{N}(m_1, m_2)$ with mean m_1 and standard deviation m_2. This allows us to compute

$$\mathbb{E}[e^{Je^{-\alpha(T-s)}}] = \exp\Big(m_1 e^{-\alpha(T-s)} + \frac{m_2^2}{2}e^{-2\alpha(T-s)}\Big). \tag{14}$$

Then the forward price $g(t, Y(t))$ when the jump size J follows Normal distribution is given as

(15)

$$g(t,Y(t)) = e^{ye^{-\alpha(T-t)}} \exp\Big(\frac{\sigma^2}{4\alpha}\big(1-e^{-2\alpha(T-t)}\big)\Big) \exp\Big(\lambda\int_t^T e^{m_1e^{-\alpha(T-s)}+\frac{m_2^2}{2}e^{-2\alpha(T-s)}}\,\mathrm{d}s - \lambda(T-t)\Big).$$

2.4.2. *Laplace distribution for jumps.* Another option for the jump size distribution is a Laplace distribution (see Section 5 for the details). Besides explanatory advantages such as capturing the heavy tails spike nature, we have the following useful property: when $J \sim \mathcal{L}aplace(m_1, m_2)$ with m_1 – the location parameter and $m_2 > 0$ – the scale parameter, then $e^{cJ} \sim \mathcal{L}og\mathcal{L}aplace(cm_1, cm_2)$ with some constant c. A very detailed investigation of the LogLaplace distribution can be found in a book of Kozubowski and Podgorski (2003). We can use their formula for the expected value of a random variable e^{cJ} and obtain

$$\mathbb{E}[e^{cJ}] = \frac{\delta}{1-c^2m_2^2}, \tag{16}$$

where $\delta := e^{cm_1}$. Then this allows us to compute the expected value

$$\mathbb{E}[e^{Je^{-\alpha(T-s)}}] = \frac{\exp(e^{-\alpha(T-s)}m_1)}{1 - e^{-2\alpha(T-s)}m_2^2}. \tag{17}$$

So the the forward price $g(t, Y(t))$ when the jump size J follows Laplace distribution is given as

(18)

$$g(t, Y(t)) = e^{ye^{-\alpha(T-t)}} \exp\Big(\frac{\sigma^2}{4\alpha}\big(1-e^{-2\alpha(T-t)}\big)\Big) \exp\Big(\lambda \int_t^T \Big(\frac{\exp(e^{-\alpha(T-s)}m_1)}{1 - e^{-2\alpha(T-s)}m_2^2}\Big)\,\mathrm{d}s - \lambda(T-t)\Big).$$

2.5. **Forward price of the threshold model.** To get the forward price $f(t, X(t))$ with the process $X(t)$ given in Equation (6) for the threshold model we start with calculating the generator of the jump-diffusion process $X(t)$ for the threshold model. First, we know that at time $t = T$ the forward price $F(T, T, X(T)) = S_T$ by the definition of the forward. We can describe $f(t, X(t))$ by the backward Kolmogorov equation

$$f_t + \mathcal{L}f = 0, \text{ for } t < T, \text{ with } f(T, x) = e^x \text{ at } t = T, \tag{19}$$

where

$$\mathcal{L}f = -\alpha x f_x + \frac{\sigma^2}{2} f_{xx} + \lambda h(x)\mathbb{E}\Big[f(x + J, t) - f(x, t)\Big], \tag{20}$$

where the expected value in the last term is over the probability distribution of jumps. Let us show how to get this backward integro-PDE. Knowing that the compensated Poisson process $\tilde{N}_t = N_t - \lambda t$ is a martingale, we start with applying Ito's formula to $f(t, X(t))$ which gives

$$\begin{aligned}
f(T, X(T)) - f(t, x) &= \int_t^T (\sigma f_x)(s, X(s))\,\mathrm{d}W_s + \int_t^T \big(f_s - \alpha x f_x + \frac{\sigma^2}{2} f_{xx}\big)(s, X(s))\,\mathrm{d}s \\
&+ \int_t^T h(X(s))\,\big[f(s, X(s) + J) - f(s, X(s))\big]\,\mathrm{d}N_s \\
&= \int_t^T (\sigma f_x)(s, X(s))\,\mathrm{d}W_s + \int_t^T \big(f_s - \alpha x f_x + \frac{\sigma^2}{2} f_{xx}\big)(s, X(s))\,\mathrm{d}s \\
&+ \int_t^T \lambda\, h(X(s))\,\big[f(s, X(s) + J) - f(s, X(s))\big]\,\mathrm{d}s \\
&+ \int_t^T h(X(s))\,\big[f(s, X(s) + J) - f(s, X(s))\big]\,\mathrm{d}\tilde{N}_s.
\end{aligned}$$

Now let us take the expected values from both sides and divide by $T - t$ which yields

$$\begin{aligned} &\frac{\mathbb{E}[f(T,X(T))]-f(t,x)}{T-t} \\ = &\frac{\int_t^T \left\{\left(f_s-\alpha x f_x+\frac{\sigma^2}{2}f_{xx}\right)(s,X(s))+\lambda\,\mathbb{E}\Big[h(X(s))\left(f(s,X(s)+J)-f(s,X(s))\right)\Big]\right\}\mathrm{d}s}{T-t}, \end{aligned}$$

then take the limit with $T-t\to 0$ and obtain

$$0=f_t-\alpha x f_x+\frac{\sigma^2}{2}f_{xx}+\lambda h(x)\mathbb{E}[f(t,x+J)-f(t,x)], \tag{21}$$

since we need that $\mathbb{E}[f(T,X(T))\,|\,X(t)=x]=f(t,X(t))$ to be fulfilled. This is exactly Equation (19).

In the next section we will implement the numerical scheme that solves this partial differential equation with an integral term to investigate the property of the function $h(x)$ on the forward price.

Before to continue with the numerical investigation, we make one more clarifying step. Let us rewrite the term of the expected jump in Equation (20) as

$$\begin{aligned} \mathbb{E}[f(t,x+J)-f(t,x)] &= \int_{-\infty}^{\infty}\big(f(t,x+y)-f(t,x)\big)\,f_Y(y)\ \mathrm{d}y \\ &\approxeq \int_{-\infty}^{\infty}\frac{\big(f(t,x+y)-f(t,x)\big)}{\mathrm{d}x}\,y\,f_Y(y)\ \mathrm{d}y \\ &= f_x\underbrace{\int_{-\infty}^{\infty}y\,f_Y(y)\ \mathrm{d}y}_{\approxeq C}. \end{aligned} \tag{22}$$

with $f_Y(y)$ is a probability density function of the random jump size J. We mean here that the integral part in this equation behaves like a gradient term, approximately giving a rise to a second order differential operator as the right-hand side of Equation (23). So then Equation (21) can be re-written as

$$f_t\approxeq(-\alpha x+\lambda h(x)C)\,f_x+\frac{\sigma^2}{2}f_{xx}, \tag{23}$$

which clearly indicates that the term in front of f_x is of discontinuous nature. It also points out the curvature due to $h(x)$ function. This "discontinuous" curvature can clearly be seen in Figure 1.

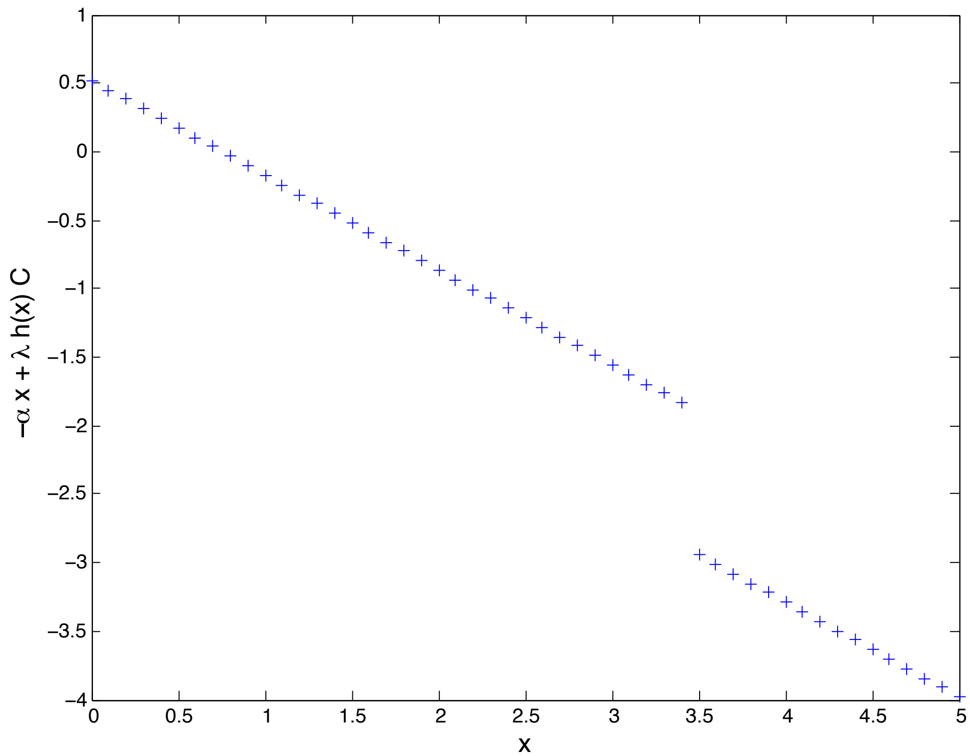


Figure 1. The curvature due to $h(x)$ function. Parameters: $C = 10$, $\alpha = 0.69$ (more than half a day), $\sigma = 2.59$, $\lambda = 13.5$ spikes per year, $\mathcal{T} = 3.5$.

3. Numerical implementation

This section contains three parts. Firstly, we discuss the method of finite difference which we apply to solve the integro-pde. When solving this equation we have to restrict our domain for the values x. It results in the truncation error which we discuss in the the second part of this section. Finally, we have to cut the integral term and investigate this truncation error as well.

3.1. **Method.**

In this section we implement the finite difference method to solve equation (19). Excellent study of numerical methods in application to finance is given in the books of Cont and Tankov (2004) and Fusai and Roncoroni (2008).This scheme is an approximation for this equation and is based on replacing derivatives by finite differences in the equation. The integral term responsible for the jumps is approximated by the Riemann sums. Since we have time and space variables we have to discretise the time and spacial domains.

Let us first re-write the Equation in (19) for the forward price $f(t, x)$ we solve via finite difference method

$$f_t = \alpha x f_x - \frac{\sigma^2}{2} f_{xx} + \lambda h(x) f - \lambda h(x) \int_{-\infty}^{\infty} f(t, x+y)\, f_Y(y)\, \mathrm{d}y, \quad f(T,x) = e^x. \tag{24}$$

Then let us make the following replacement: $T - t = \tau$, which allows us to move backward in time when solving the equation numerically. Then the integro-pde becomes

$$f_\tau = -\alpha x f_x + \frac{\sigma^2}{2} f_{xx} - \lambda h(x) f + \lambda h(x) \int_{-\infty}^{\infty} f(T-\tau, x+y)\, f_Y(y)\, \mathrm{d}y, \quad f(\tau,x) = e^x. \tag{25}$$

Since there is enormous variety of sources on the finite difference method precisely applied to solving financial mathematics problems, we will not focus on the details in general. Instead, we provide the exact scheme we use here to solve Equation (25).

We start with a time domain for $\tau \in [0, T]$ and discretise it with $\Delta\tau = \frac{\tau}{N}$ with N being the number of time steps. Then we continue with the space domain (x_{min}, x_{max}) and $\Delta x = \frac{x_{max} - x_{min}}{M}$ with M being the number of space steps. We also need to introduce some values K_{min} and K_{max} responsible for the interval for the jump size. The time and space derivatives become

$$\frac{\partial f}{\partial \tau} \approxeq \frac{f^{n+1} - f^n}{\Delta\tau}, \tag{26}$$

$$\frac{\partial f}{\partial x} \approxeq \frac{f^{n+1}_{i+1} - f^{n+1}_{i}}{\Delta x}, \tag{27}$$

$$\frac{\partial^2 f}{\partial x^2} \approxeq \frac{f^{n+1}_{i+1} - 2f^{n+1}_{i} + f^{n+1}_{i-1}}{(\Delta x)^2}, \tag{28}$$

with $n = 1, \ldots, N+1$ and $i = 1, \ldots, M+1$. Here we use a so-called explicit scheme. Further we need to approximate the integral term. We do this via trapezoidal quadrature rule with the same grid resolution Δx. As it is stated in Cont and Tankov (2004), due to computational complexity when there is a jump term, it is more convenient to use implicit scheme for the integral part

$$\begin{aligned}
\int_{-\infty}^{\infty} f(T-\tau, x_i+y)\, f_Y(y)\, \mathrm{d}y &\approxeq \lim_{\substack{K_{min}\to-\infty \\ K_{max}\to\infty}} \int_{K_{min}}^{K_{max}} f(T-\tau, x_i+y)\, f_Y(y)\, \mathrm{d}y \\
&\approxeq \sum_{j=K_l}^{K_u} f^n(x_i + y_j) \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} f_Y(y)\, \mathrm{d}y \\
&= \sum_{j=K_l}^{K_u} f^n_{i+j} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} f_Y(y)\, \mathrm{d}y,
\end{aligned} \tag{29}$$

where $[K_{min}, K_{max}] \in [(K_l - 1/2)\Delta x, (K_l + 1/2)\Delta x]$.

So, the total scheme then becomes explicit-implicit, namely

$$\frac{f^{n+1} - f^n}{\Delta t} = Df^{n+1} + Jf^n, \tag{30}$$

where

$$(Df^{n+1})_i = \alpha x_i \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x} - \frac{\sigma^2}{2} \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2}, \tag{31}$$

$$(Jf^n)_i = -\lambda h(x_i) \sum_{j=K_l}^{K_u} f_{i+j}^n \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} f_Y(y)\, \mathrm{d}y. \tag{32}$$

This results in

$$(I - \Delta\tau D) f^{n+1} = (I + \Delta\tau J) f^n, \tag{33}$$

where I is identity matrix, $(I - \Delta\tau D)$ and $(I + \Delta\tau J)$ are tridiagonal matrices. Since we know the final condition, i.e. $f(\tau, x) = e^x$ we move backward in time when solving this scheme. In other words, we know the f^n and search for the f^{n+1}.

As the reader can see, there are two things we should agree on when solving this equation: when $i = 1$ we need to know the term f_{i-1}^{n+1} which is out of the domain and when $i = M + 1$ we need to know the term f_{i+1}^{n+1} which is also out of the domain. There are several possible solutions to this obstacle. One of them is to represent the second derivative differently and then not to go out of the domain of (x_{min}, x_{max}). Another opportunity is to assume that our final condition at $\tau = 0$ is extended not only for the domain $[x_{min}, x_{max}]$ but for all the values x we need. However, in our case we can go for the third option and find these values explicitly since we have an exact solution for the forward price for the classical model considered above, at least for some distributions. We mean here the following: when $x > x_{max}$ it implies that the function $h(x) = -1$, which gives forward price $f(t, X(t))$ is almost equal to $g(t, Y(t))$: the difference is the minus sign in front of the jump component in Equation (5) (in the next section we show that if the jump size distribution is symmetric, then $f(t, X(t))$ is equal to $g(t, Y(t))$ when $x > x_{max}$ and $h(x) = -1$). When $x < x_{min}$ it implies that the function $h(x) = 1$, which gives forward price $f(t, X(t))$ is exactly equal to $g(t, Y(t))$.

There are two important points needed to be clarified when dealing with the finite difference scheme: consistence with the continuous equation and stability (i.e. the found solution does not blow up when $\Delta\tau$ and Δx goes to 0). The choice of those should be done carefully. The stability condition stated in Cont and Tankov (2004) and used here is $\Delta\tau \leq \inf\{\frac{1}{\lambda}, \frac{(\Delta x)^2}{\sigma^2}\}$.

3.2. **Domain truncation error.** When we solve numerically the partial differential equation with an integral part, we have to define the domain for the x values. Defining this interval for $x \in (x_{min}, x_{max})$ means that we have to specify some boundary conditions at $x = x_{min}$ and $x = x_{max}$. On top of that we have specify the boundary values for our function for the integral term. In our case we can specify these boundary values explicitly. When $x \leq x_{min}$, then the function $h(x) = 1$ and then the forward price $f(t, X(t)) = g(t, Y(t))$. When $x \geq x_{max}$, the function $h(x) = -1$, then the forward price $f(t, X(t)) = g^{-}(t, Y(t))$ with minus in front of the jump component in Equation (5), namely

$$dY^{-}(t) = -\alpha Y(t)\ \mathrm{d}t + \sigma\ \mathrm{d}W_t -\ \mathrm{d}Q_t. \tag{34}$$

Let us now compute the solution associated with this process $Y^{-}(t)$ assuming that the starting value y is the same as for the function $g(t, y)$

$$g^{-}(t, y) = e^{ye^{-\alpha(T-t)}} \exp\Big(\frac{\sigma^2}{4\alpha}\big(1 - e^{-2\alpha(T-t)}\big)\Big) \exp\Big(\lambda \int_t^T (\mathbb{E}[e^{-Je^{-\alpha(T-s)}}] - 1)\, \mathrm{d}s\Big). \tag{35}$$

We can see that the only difference between $g(t, y)$ and $g^{-}(t, y)$ lies in the expected values of the exponent of the jump component, let us compare these terms keeping in mind that $e^{-\alpha(T-s)} \in (0, 1]$ for $\alpha > 0$ and $T - s \geq 0$

$$\begin{aligned}
&\mathbb{E}[e^{Je^{-\alpha(T-s)}} - e^{-Je^{-\alpha(T-s)}}] \\
=\ & \int_{-\infty}^{+\infty} \Big(e^{xe^{-\alpha(T-s)}} - e^{-xe^{-\alpha(T-s)}}\Big)\ f_X(x)\ \mathrm{d}x \\
=\ & \int_{-\infty}^{0} \Big(e^{xe^{-\alpha(T-s)}} - e^{-xe^{-\alpha(T-s)}}\Big)\ f_X(x)\ \mathrm{d}x + \int_0^{+\infty} \Big(e^{xe^{-\alpha(T-s)}} - e^{-xe^{-\alpha(T-s)}}\Big)\ f_X(x)\ \mathrm{d}x \\
=\ & \int_0^{+\infty} \Big(e^{-xe^{-\alpha(T-s)}} - e^{xe^{-\alpha(T-s)}}\Big)\ f_X(-x)\ \mathrm{d}x + \int_0^{+\infty} \Big(e^{xe^{-\alpha(T-s)}} - e^{-xe^{-\alpha(T-s)}}\Big)\ f_X(x)\ \mathrm{d}x \\
=\ & \begin{cases} 0, & \text{if distribution is symmetric} \\ \int_0^{+\infty} \Big(e^{xe^{-\alpha(T-s)}} - e^{-xe^{-\alpha(T-s)}}\Big)\ \big(f_X(x) - f_X(-x)\big)\ \mathrm{d}x, & \text{otherwise} \end{cases} \\
=\ & \begin{cases} 0, & \text{if} \quad f_X(x) = f_X(-x), & \text{distribution is symmetric;} \\ C_{>0}, & \text{if} \quad f_X(x) > f_X(-x), & \text{upward jumps are more likely;} \\ C_{<0}, & \text{if} \quad f_X(x) < f_X(-x), & \text{downward jumps are more likely.} \end{cases}
\end{aligned} \tag{36}$$

So we observe that

$$\begin{cases} g(t, y) = g^{-}(t, y), & \text{distribution is symmetric;} \\ g(t, y) > g^{-}(t, y), & \text{upward jumps are more likely;} \\ g(t, y) < g^{-}(t, y), & \text{downward jumps are more likely;} \end{cases}$$

Let us then denote the solution to our boundary problem given in Equation (19) as

$$f_B(t,x) = \begin{cases} \hat{f}_B(t,x), & \text{if } x \in (x_{min}, x_{max}) \\ g(t,y), & \text{if } x \leq x_{min} \\ g^-(t,y), & \text{if } x \geq x_{max} \end{cases} \tag{37}$$

where $\hat{f}_B(t,x)$ solves Equation (24) on a bounded domain (x_{min}, x_{max}), solution $g(t,y)$ is given in Equation (18) and solution $g^-(t,y)$ is given in Equation (35). Now we are ready to calculate the domain truncation error with the following proposition.

Proposition 3.1. *Assume that*

- *for $\nu(\,\mathrm{d}x)$ being a Lévy measure and for $\epsilon > 0$, $\int_{|x|>1} \left(e^{\epsilon|x|} - 1\right) \nu(\,\mathrm{d}x) < \infty$;*
- $x_{min} = -x_{max}$.

Let $f(t,x)$ be the solution of our problem in Equation (19) on the unbounded region and $f_B(t,x)$ be the solution defined in Equation (37). Then

$$|f(t,x) - f_B(t,x)| \leq 2\,C\,e^{-\epsilon\,(x_{max} - |\tilde{x}|}\begin{cases} g(t,y), & \text{if} \quad f_X(x) \geq f_X(-x), \\ g^-(t,y), & \text{if} \quad f_X(x) < f_X(-x), \end{cases}$$

where C is some constant and does not depend on x_{max} and $f_X(\cdot)$ is a probability density for a jump size distribution.

Proof. For $t < T$ denote by

$$Z(T) := \sigma \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)} h(X(s))\,\mathrm{d}Q_s,$$

then $X(T) = xe^{-\alpha(T-t)} + Z(T)$. Denote also by $M_T^x := \sup_{t \leq s \leq T} |X(s)|$. Then for $\tau := \inf\{s \geq t \,|\, M_s^x \geq x_{max}\}$ being the first time when the process $X(t)$ leaves the interval (x_{min}, x_{max}) we have

$$\begin{aligned}
& |f(t,x) - f_B(t,x)| \\
= \; & \left|\mathbb{E}\left[e^{X(T)}\right] - \mathbb{E}\left[e^{X(T)}\mathbf{1}_{M_T^x < x_{max}}\right] - \mathbb{E}\left[e^{X(\tau)}\mathbf{1}_{M_T^x \geq x_{max}}\right]\right| \\
= \; & \left|\mathbb{E}\left[e^{X(T)}\mathbf{1}_{M_T^x \geq x_{max}}\right] - \mathbb{E}\left[e^{X(\tau)}\mathbf{1}_{M_T^x \geq x_{max}}\right]\right| \\
\leq \; & 2\,\mathbb{E}\left[\left|e^{X(T)}\mathbf{1}_{M_T^x \geq x_{max}}\right|\right] \\
= \; & 2\left(g^-(t,y)\,\mathbb{P}\left(\sup_{t \leq s \leq T} X(s) \geq x_{max}\right) + g(t,y)\,\mathbb{P}\left(\sup_{t \leq s \leq T}(-X(s)) \geq x_{max}\right)\right) \\
= \; & 2\left(g^-(t,y)\,\mathbb{P}\left(\sup_{t \leq s \leq T} X(s) \geq x_{max}\right) + g(t,y)\,\mathbb{P}\left(-\inf_{t \leq s \leq T} X(s) \geq x_{max}\right)\right) \\
= \; & 2\left(g^-(t,y)\,\mathbb{P}\left(\sup_{t \leq s \leq T} X(s) \geq x_{max}\right) + g(t,y)\,\mathbb{P}\left(\inf_{t \leq s \leq T} X(s) \leq x_{min}\right)\right)
\end{aligned}$$

$$\leq \begin{cases} 2\,g(t,y)\,\left(\mathbb{P}\left(\sup_{t\leq s\leq T} X(s)\geq x_{max}\right)+\mathbb{P}\left(\inf_{t\leq s\leq T} X(s)\leq x_{min}\right)\right), & \text{if} \quad f_X(x)\geq f_X(-x), \\ 2\,g^{-}(t,y)\,\left(\mathbb{P}\left(\sup_{t\leq s\leq T} X(s)\geq x_{max}\right)+\mathbb{P}\left(\inf_{t\leq s\leq T} X(s)\leq x_{min}\right)\right), & \text{if} \quad f_X(x)< f_X(-x), \end{cases}$$
$$= \begin{cases} 2\,g(t,y)\,\mathbb{P}\left(M_T^x\geq x_{max}\right), & \text{if} \quad f_X(x)\geq f_X(-x), \\ 2\,g^{-}(t,y)\,\mathbb{P}\left(M_T^x\geq x_{max}\right), & \text{if} \quad f_X(x)< f_X(-x). \end{cases} \tag{38}$$

Since the function $h(x)$ is bounded by -1 and 1, we observe that

$$\begin{aligned} Z(T) &= \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s+\int_t^T e^{-\alpha(T-s)}h(X(s))\,\mathrm{d}Q_s \\ &\leq \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s+\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s \\ &=: \tilde{Z}^T, \end{aligned} \tag{39}$$

where $\tilde{Q}_t$ is a compound Poisson process and $\tilde{Z}^T$ is a Lévy process with a measure $\nu(\mathrm{d}x)$.

$$M_T^x=\sup_{t\leq s\leq T}\left|xe^{-\alpha(s-t)}+Z(s)\right|\leq \sup_{t\leq s\leq T}\left|\underbrace{xe^{-\alpha(s-t)}}_{=:\tilde{x}}+\tilde{Z}(s)\right| =: M_T^{\tilde{x}}. \tag{40}$$

We note that

$$\mathbb{P}\left(M_T^x\geq x_{max}\right)\leq \mathbb{P}\left(M_T^{\tilde{x}}\geq x_{max}\right). \tag{41}$$

Now with the help of Theorem 2 we find that

$$C:=\mathbb{E}e^{\epsilon\,M_T^{\tilde{0}}}<\infty. \tag{42}$$

Applying further Chebyshev's inequality gives

$$\mathbb{P}\left(M_T^{\tilde{0}}\geq x_{max}\right)\leq Ce^{-\epsilon\,x_{max}}. \tag{43}$$

Now we show that

$$\begin{aligned} \mathbb{P}\left(M_T^{\tilde{x}}\geq x_{max}\right) &= \mathbb{P}\left(\sup_{t\leq s\leq T}\left|\tilde{x}+\tilde{Z}(s)\right|\geq x_{max}\right) \\ &\leq \mathbb{P}\left(\sup_{t\leq s\leq T}\left|\tilde{Z}(s)\right|+|\tilde{x}|\geq x_{max}\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left(\sup_{t\le s\le T}\left|\tilde{Z}(s)\right| \ge x_{max}-|\tilde{x}|\right)\\
&= \mathbb{P}\left(M_T^{\tilde{0}} \ge x_{max}-|\tilde{x}|\right)\\
&\le Ce^{-\epsilon\,(x_{max}-|\tilde{x}|}. \qquad (44)
\end{aligned}
$$

□

3.3. **Jump size domain truncation error.** Here we discuss the integral term truncation error which we obtain when we cut the interval for the jump size as it was done in Equation (29). Process Q_t responsible for the jump component in Equation (6) is a compound Poisson process with a jump measure $\nu(\,\mathrm{d}x)$ that measures the expected number of jumps per unit time whose size belong to a set $A \in \mathcal{B}(\mathbb{R})$.

Now let us introduce a new compound Poisson process Q_t^K with a new jump measure $\nu^K := \nu(\,\mathrm{d}x)\,1_{x\in[K_{min},K_{max}]}$. Then we show the relation between the process $X(T)$ (with the jump component formed by the process Q_t) for some $t \le T$ assuming that $X(t) = x$ at time t and having that function $h(x)$ is bounded by -1 and 1

$$
\begin{aligned}
X(T) &= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)}h(X(s))\,\mathrm{d}Q_s\\
&= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \sum_{t<s\le T} e^{-\alpha(T-s)}h(X(s))\Delta Q(s)\\
&= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \sum_{t<s\le T} e^{-\alpha(T-s)}h(X(s))\sum_{i=N(t)+1}^{N(s)} J_i\\
&= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \sum_{t<s\le T} e^{-\alpha(T-s)}h(X(s))J\\
&\le xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \sum_{t<s\le T} e^{-\alpha(T-s)}\,|J|\\
&= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \sum_{t<s\le T} e^{-\alpha(T-s)}\sum_{i=N(t)+1}^{N(s)} |J|\\
&= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \sum_{t<s\le T} e^{-\alpha(T-s)}\Delta\tilde{Q}(s)\\
&= xe^{-\alpha(T-t)} + \sigma\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s, \qquad (45)
\end{aligned}
$$

where in the third equality we used a definition of a compound Poisson process given in Section 11.3.1 in Shreve (2004). Analogously, the process $X^K(T)$ (with the jump component formed by the process Q_t^K) with $X^K(t) = x$ at time t is

$$
\begin{aligned}
X^K(T) &= xe^{-\alpha(T-t)} + \sigma \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)} h(X^K(s))\,\mathrm{d}Q_s^K \\
&\leq xe^{-\alpha(T-t)} + \sigma \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}W_s + \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s^K, \qquad (46)
\end{aligned}
$$

where J^K is a random variable responsible for the jump size and that falls into the interval $I = [K_{min}, K_{max}]$.

Let us now introduce a solution $f_I(t,x)$ which solves the problem given in Equation (19) but with the truncated jump size domain $[K_{min}, K_{max}]$ and a solution $f(t,x)$ with an unbounded jump size domain $(-\infty, +\infty)$. We give an error estimate associated to this truncation. This error estimate derivation is similar to the proof of Proposition 4.2 in Cont and Voltchkova (2005). The main difference here is the presence of function $h(x)$ due to which our process $X(t)$ is not a Lévy process.

Proposition 3.2. *Assume that a jump measure $\nu(\,\mathrm{d}x)$ for a compound Poisson process satisfies the following*

- $\int_{\mathbb{R}} \nu(\,\mathrm{d}x) < \infty$;
- $K_{max} > 1$ *and* $K_{min} = -K_{max}$;
- *for* $\xi_1, \xi_2 > 0$, $\int_{-\infty}^{-1} |x|\, e^{\xi_1 |x|}\, \nu(\,\mathrm{d}x) < \infty$ *and* $\int_1^{\infty} |x|\, e^{\xi_2 |x|}\, \nu(\,\mathrm{d}x) < \infty$;
- *for* $\epsilon > 0$, $\int_{\mathbb{R}} \left(e^{\epsilon |x|} - 1\right) \nu(\,\mathrm{d}x) < \infty$ *(for the details see Cont and Tankov (2004), Proposition 3.8,).*

Then

$$
\begin{aligned}
& |f(t,x) - f_I(t,x)| \\
\leq\; & e^{C_1^{t,T} + \frac{1}{2} C_1^{t,T}}\, e^{(T-t)\left(C_3\, e^{-\beta_1 |K_{min}|} + C_4\, e^{-\beta_2 |K_{max}|}\right)} \Bigg(e^{(T-t)\left(C_5\, e^{-\beta_1 |K_{min}|} + C_6\, e^{-\beta_2 |K_{max}|}\right)} - 1 \\
+\; & 2\left(1 - e^{-\alpha(T-t)}\right)\left(C_7\, e^{-\beta_1 |K_{min}|} + C_8\, e^{-\beta_2 |K_{max}|}\right)\Bigg), \qquad (47)
\end{aligned}
$$

where $C_1^{t,T} = xe^{-\alpha(T-t)}$, $C_2^{t,T} = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-t)})$, *some constants* $\beta_1, \beta_2 > 0$ *and come constants* C_3, C_4, C_5, C_6, C_7 *and* C_8.

Proof. We recall a few useful facts and give some notations:

- $|e^x - 1| = (e^x - 1) + 2\,(1 - e^x)^+ \leq (e^x - 1) + 2\,|x|$;
- by $U_T := xe^{-\alpha(T-t)} + \int_t^T \sigma^{-\alpha(T-s)}\,\mathrm{d}W_s$ we denote a process that follows Gaussian distribution with mean $C_1^{t,T}$ and variance $C_2^{t,T}$;
- by $\tilde{Q}_t$ we denote a compound Poisson process with a measure $\tilde{\nu}(\,\mathrm{d}x) := \nu(\,\mathrm{d}x)\, 1_{|x|}$;
- by $\tilde{Q}_t^K$ we denote a compound Poisson process with a measure $\tilde{\nu}^K(\,\mathrm{d}x) := \nu(\,\mathrm{d}x)\, 1_{|x| \in [K_{min}, K_{max}]}$;

- by $D_t^K := \tilde{Q}_t - \tilde{Q}_t^K$ we denote a compound Poisson process with a measure $\hat{\nu}^K(\,\mathrm{d}x) = \tilde{\nu}(\,\mathrm{d}x) - \tilde{\nu}^K(\,\mathrm{d}x) = \nu(\,\mathrm{d}x)\,1_{|x|\notin[K_{min},K_{max}]}$;

Then we have

$$
\begin{aligned}
&|f(t,x) - f_I(t,x)| \\
=\;& \left|\mathbb{E}\left[e^{U_T + \int_t^T e^{-\alpha(T-s)} h(X(s))\,\mathrm{d}Q_s} - e^{U_T + \int_t^T e^{-\alpha(T-s)} h(X^K(s))\,\mathrm{d}Q_s^K}\right]\right| \\
\leq\;& e^{C_1^{t,T} + \frac{1}{2}C_2^{t,T}} \left|\mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s} - e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s^K}\right]\right| \\
=\;& e^{C_1^{t,T} + \frac{1}{2}C_2^{t,T}} \left|\mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s^K}\left(e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K} - 1\right)\right]\right| \\
\leq\;& e^{C_1^{t,T} + \frac{1}{2}C_2^{t,T}}\, \mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s^K}\right] \mathbb{E}\left[\left|e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K} - 1\right|\right] \\
\leq\;& e^{C_1^{t,T} + \frac{1}{2}C_2^{t,T}}\, \mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s^K}\right] \left(\mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K} - 1\right] + 2\mathbb{E}\left[\left|\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K\right|\right]\right).
\end{aligned}
\tag{48}
$$

Now let us show that every term is bounded by some constant. Since the first term is easily computable, we start with the second term

$$
\begin{aligned}
&\mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}\tilde{Q}_s^K}\right] \\
=\;& e^{\int_t^T \int_{\mathbb{R}}\left(e^{e^{-\alpha(T-s)}|x|} - 1\right)\tilde{\nu}^K(\,\mathrm{d}x)\,\mathrm{d}s} \\
=\;& e^{\int_t^T \int_{K_{min}}^0 \left(e^{e^{-\alpha(T-s)}(-x)} - 1\right)\nu(\,\mathrm{d}x)\,\mathrm{d}s + \int_t^T \int_0^{K_{max}}\left(e^{e^{-\alpha(T-s)}x} - 1\right)\nu(\,\mathrm{d}x)\,\mathrm{d}s} \\
\leq\;& e^{\int_t^T \int_{K_{min}}^0 \left(e^{-x} - 1\right)\nu(\,\mathrm{d}x)\,\mathrm{d}s + \int_t^T \int_0^{K_{max}}(e^{x} - 1)\,\nu(\,\mathrm{d}x)\,\mathrm{d}s} \\
=\;& e^{\int_t^T \int_{\mathbb{R}}\left(e^{|x|} - 1\right)\tilde{\nu}^K(\,\mathrm{d}x)\,\mathrm{d}s} \\
=\;& e^{(T-t)\left(e^{-\beta_1|K_{min}|}\int_{K_{min}}^0\left(e^{|x|+\beta_1|K_{min}|} - e^{\beta_1|K_{min}|}\right)\nu(\,\mathrm{d}x) + e^{-\beta_2|K_{max}|}\int_0^{K_{max}}\left(e^{|x|+\beta_2|K_{max}|} - e^{\beta_2|K_{max}|}\right)\nu(\,\mathrm{d}x)\right)} \\
\leq\;& e^{(T-t)\left(e^{-\beta_1|K_{min}|}\int_{K_{min}}^0\left(e^{|x|+\beta_1|K_{min}|} - 1\right)\nu(\,\mathrm{d}x) + e^{-\beta_2|K_{max}|}\int_0^{K_{max}}\left(e^{|x|+\beta_2|K_{max}|} - 1\right)\nu(\,\mathrm{d}x)\right)} \\
\leq\;& e^{(T-t)\left(C_3\, e^{-\beta_1|K_{min}|} + C_4\, e^{-\beta_2|K_{max}|}\right)}.
\end{aligned}
\tag{49}
$$

Analogously, we consider the first term in the sum in the brackets of Equation (48)

$$
\begin{aligned}
&\mathbb{E}\left[e^{\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K} - 1\right] \\
=\;& e^{\int_t^T \int_{\mathbb{R}}\left(e^{e^{-\alpha(T-s)}|x|} - 1\right)\hat{\nu}^K(\,\mathrm{d}x)\,\mathrm{d}s} - 1 \\
=\;& e^{\int_t^T \int_{-\infty}^{K_{min}}\left(e^{e^{-\alpha(T-s)}(-x)} - 1\right)\nu(\,\mathrm{d}x)\,\mathrm{d}s + \int_t^T \int_{K_{max}}^{+\infty}\left(e^{e^{-\alpha(T-s)}x} - 1\right)\nu(\,\mathrm{d}x)\,\mathrm{d}s} - 1 \\
\leq\;& e^{\int_t^T \int_{-\infty}^{K_{min}}\left(e^{-x} - 1\right)\nu(\,\mathrm{d}x)\,\mathrm{d}s + \int_t^T \int_{K_{max}}^{+\infty}(e^{x} - 1)\,\nu(\,\mathrm{d}x)\,\mathrm{d}s} - 1 \\
=\;& e^{\int_t^T \int_{\mathbb{R}}\left(e^{|x|} - 1\right)\hat{\nu}^K(\,\mathrm{d}x)\,\mathrm{d}s} - 1
\end{aligned}
$$

$$\leq \; e^{(T-t)\left(e^{-\beta_1|K_{min}|}\int_{-\infty}^{K_{min}}\left(e^{|x|+\beta_1|K_{min}|}-1\right)\nu(\,\mathrm{d}x)+e^{-\beta_2|K_{max}|}\int_{K_{max}}^{+\infty}\left(e^{|x|+\beta_2|K_{max}|}-1\right)\nu(\,\mathrm{d}x)\right)}-1$$
$$\leq \; e^{(T-t)\left(C_5\, e^{-\beta_1|K_{min}|}+C_6\, e^{-\beta_2|K_{max}|}\right)}-1. \tag{50}$$

Finally we compute the boundary for $\mathbb{E}\left[\left|\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K\right|\right]$. This process D_t^K can be represented as a sum of two compound Poisson processes: P_t^K with a measure $\nu(\,\mathrm{d}x)\,1_{x>K_{max}}$ and N_t^K with a measure $\nu(\,\mathrm{d}x)\,1_{x<K_{min}}$. More precisely, the process $P_t^K \geq 0$ has only positive jumps not smaller than $K_{max} > 1$ and the process $N_t^K \leq 0$ has only negative jumps not greater than $K_{min} < -1$. Then we have

$$\begin{aligned}
& \mathbb{E}\left[\left|\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}D_s^K\right|\right] \\
= \; & \mathbb{E}\left[\left|\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}P_s^K + \int_t^T e^{-\alpha(T-s)}\,\mathrm{d}N_s^K\right|\right] \\
\leq \; & \mathbb{E}\left[\left|\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}P_s^K\right|\right] + \mathbb{E}\left[\left|\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}N_s^K\right|\right] \\
= \; & \mathbb{E}\left[\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}P_s^K\right] - \mathbb{E}\left[\int_t^T e^{-\alpha(T-s)}\,\mathrm{d}N_s^K\right] \\
\leq \; & \int_t^T e^{-\alpha(T-s)}\int_{-\infty}^{K_{min}}|x|\;\nu(\,\mathrm{d}x)\;\mathrm{d}s + \int_t^T e^{-\alpha(T-s)}\int_{K_{max}}^{+\infty}|x|\;\nu(\,\mathrm{d}x)\;\mathrm{d}s \\
\leq \; & \left(1-e^{-\alpha(T-t)}\right)\left(e^{-\beta_1|K_{min}|}\int_{-\infty}^{K_{min}}|x|\,e^{\beta_1|K_{min}|}\,\nu(\,\mathrm{d}x) + e^{-\beta_2|K_{max}|}\int_{K_{max}}^{+\infty}|x|\,e^{\beta_2|K_{max}|}\,\nu(\,\mathrm{d}x)\right) \\
\leq \; & \left(1-e^{-\alpha(T-t)}\right)\left(C_7\, e^{-\beta_1|K_{min}|} + C_8\, e^{-\beta_2|K_{max}|}\right).
\end{aligned} \tag{51}$$

□

4. Results and discussion

4.1. **Threshold and Cartea model forwards.** This section demonstrates the results of two models. We plot the forward prices for the threshold and for the classical model. For both models for the sake of simplicity we assume zero seasonal component $\mu(t)$. We take some calibrated parameters obtained from the German spot power market (see papers Benth, Kiesel, and Nazarova (2012) and Bannoer, Kiesel, Nazarova, and Scherer (2013)). Table 1 contains all the estimated parameters we use.

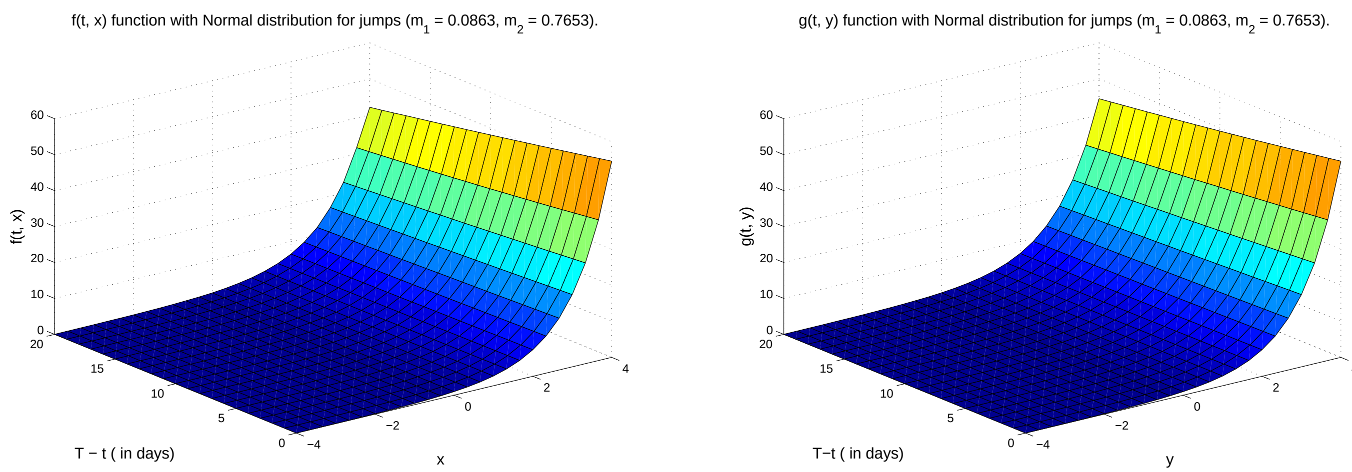
Here we provide the forward prices $f(t, X(t))$ as a solution to Equation (25) together with the forward prices given in Equation (15).

We start with assumption that jump size random variables (log scale) follow Gaussian distribution. Figure 4 shows prices $f(t, X(t))$ and $g(t, Y(t))$. When we are at the maturity, i.e. $T-t=0$, we observe our boundary condition. We also can see that the prices decrease

Table 1. An overview over the estimated parameter values for the forward price.

Parameter	Interpretation	Estimated value	Measure unit
α	mean-reversion force	0.6923	approx. 1 day
σ	volatility	2.59	
λ	jump intensity	13.5	spikes per year
$\mathcal{T}$	jump threshold	3	log scale
K_{max}	jump truncation	8	log scale

when time to maturity increases. We observe that both models produce similar results, in general the price level is almost the same. However, the threshold model produces slightly lower prices compared to the classical approach. This is perfectly in line with the behavior of function $h(x)$.



(A) Forward function $f(t, X(t))$.

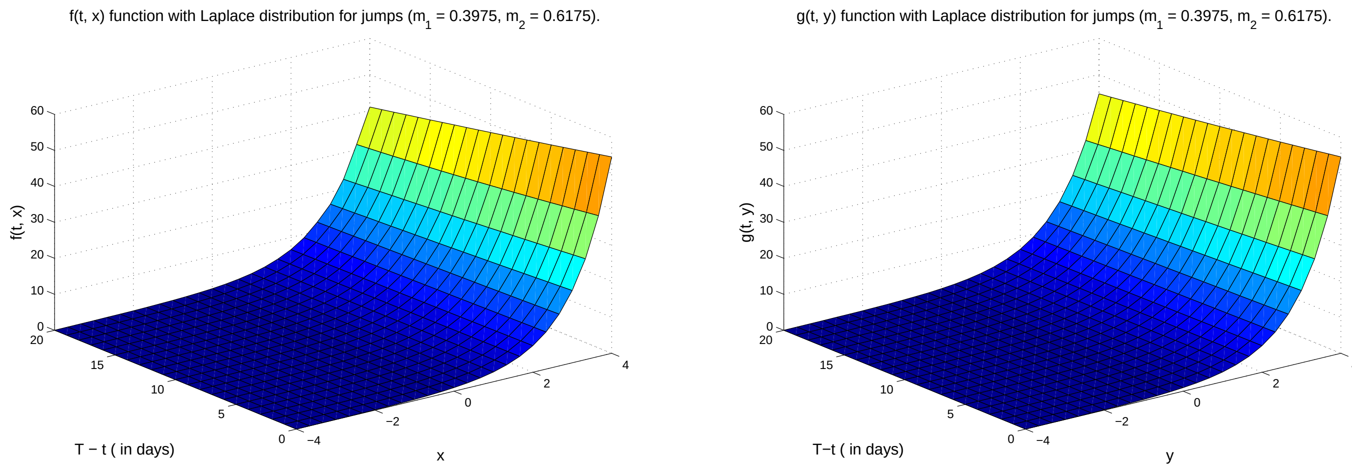
(B) Forward function $g(t, Y(t))$.

Figure 2. Normal distribution assumed for jumps with mean $m_1 = 0.0863$ and standard deviation $m_2 = 0.7653$. Time to maturity $T - t = 20$ days.

We continue with assumption that jump size random variables (log scale) follow Laplace distribution. Figure 5 shows prices $f(t, X(t))$ and $g(t, Y(t))$. When we are at the maturity, i.e. $T - t = 0$, we again observe our boundary condition fulfilled. We also can see that the prices decrease when time to maturity increases. Again both models demonstrate similar results. Moreover, different jump size distribution does not provide a significant difference to the price level. Again, the threshold model produces slightly lower prices compared to the classical approach due to the function $h(x)$.

BIG AVERAGE JUMP SIZE

4.2. **Discussion.** The function h does not show a remarkable impact on the forward prices. The prices are slightly smaller compared to those from the classical model.



(A) Forward function $f(t, X(t))$ in the log scale. (B) Forward function $g(t, Y(t))$ in the log scale.

Figure 3. Laplace distribution assumed for jumps with location $m_1 = 0.3975$ and scale $m_2 = 0.6175$. Time to maturity $T - t = 20$ days.

Table 2. Comparative final forward values for the threshold ($\mathcal{T} = 3$) and jump-diffusion models for various starting values x and various average jump size parameter m_1 values. $\tau = 20$ days to maturity.

		x = y = 4		x = y = 2	
		$f(\tau, x)$,	$g(\tau, y)$	$f(\tau, x)$	$g(\tau, y)$
Normal	$m_1 = 0.0863$	42.0727	44.4889	6.5248	6.5248
	$m_1 = 0.27$	42.0345	44.5389	6.5254	6.5248
Laplace	$m_1 = 0.3975$	40.9618	44.6442	6.6176	6.5248
	$m_1 = 1.2$	37.9248	45.1348	6.8678	6.5248

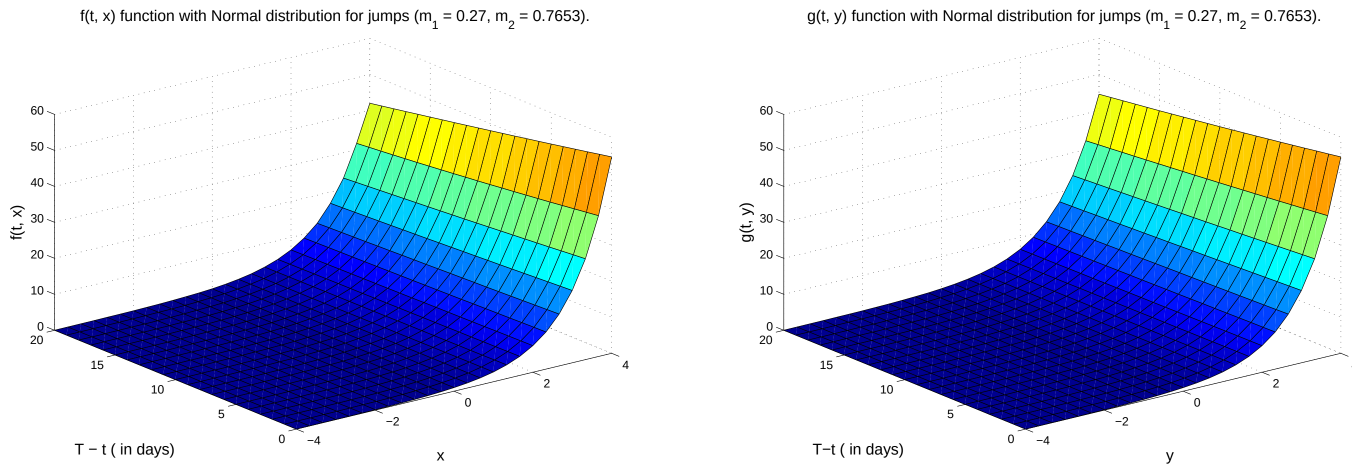
References

Bannoer, K., R. Kiesel, A. Nazarova, and M. Scherer, 2013, Model Risk and Power Plant Valuation, *Submitted to Energy Economics*.

Benth, F. E., R. Kiesel, and A. Nazarova, 2012, A critical empirical study of three electricity spot price models, *Energy Economics* 34, 1589–1616.

Cartea, A., and M. G. Figueroa, 2005, Pricing in electricity markets: a mean reverting jump diffusion model with seasonality, *Applied Mathematical Finance* 12, 313–335.

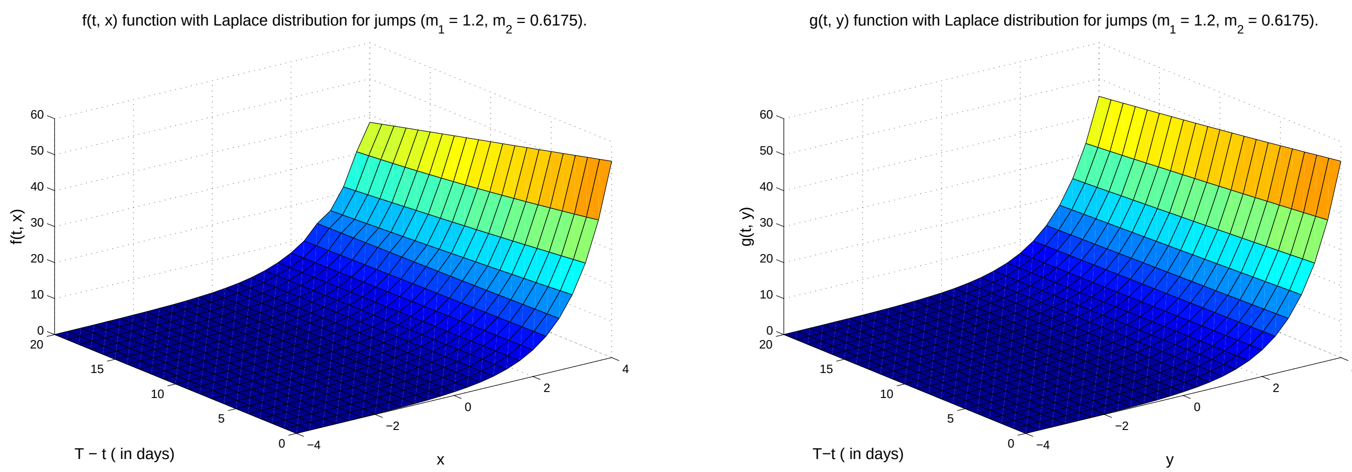
Cont, R., and P. Tankov, 2004, *Financial modelling with jump processes* vol. 133. (Chapman & Hall/CRC Boca Raton).



(A) Forward function $f(t, X(t))$.

(B) Forward function $g(t, Y(t))$.

FIGURE 4. Normal distribution assumed for jumps with mean $m_1 = 0.27$ and standard deviation $m_2 = 0.7653$. Time to maturity $T - t = 20$ days.



(A) Forward function $f(t, X(t))$ in the log scale.

(B) Forward function $g(t, Y(t))$ in the log scale.

FIGURE 5. Laplace distribution assumed for jumps with location $m_1 = 1.2$ and scale $m_2 = 0.6175$. Time to maturity $T - t = 20$ days.

Cont, R., and E. Voltchkova, 2005, A finite difference scheme for option pricing in jump diffusion and exponential Lévy models, *SIAM Journal on Numerical Analysis* 43, 1596–1626.

Fusai, G., and A. Roncoroni, 2008, *Implementing models in quantitative finance: methods and cases.* (Springer).

Geman, H., and A. Roncoroni, 2006, Understanding the Fine Structure of Electricity Prices, *The Journal of Business* 79, 1225–1261.

Kallsen, J., and A. N. Shiryaev, 2002, The cumulant process and Esscher's change of measure, *Finance and Stochastics* 6, 397–428.

Kozubowski, T., and K. Podgorski, 2003, A log-Laplace growth rate model, *The Mathematical Scientist* 28, 49–60.
Roncoroni, A., 2002, Essays in quantitative finance: Modelling and calibration in interest rate and electricity markets, *PhD diss., University Paris Dauphine*.
Sato, K.I., 1999, *Lévy processes and infinitely divisible distributions*. (Cambridge Univ Pr).
Shreve, S. E., 2004, *Stochastic calculus for finance II: Continuous-Time Modelsl* vol. 2. (Springer).

(Fred Espen Benth), Centre of Mathematics for Applications, University of Oslo, PO Box 1053 Blindern, N-0316 Oslo, Norway,

E-mail address: fredb@math.uio.no

(Rüdiger Kiesel), Chair for Energy Trading and Finance, University Duisburg-Essen, Campus Essen, Universitätsstrasse 12, 45141 Essen, Germany,

E-mail address: Ruediger.Kiesel@uni-due.de

(Anna Nazarova), Chair for Energy Trading and Finance, University Duisburg-Essen, Campus Essen, Universitätsstrasse 12, 45141 Essen, Germany,

E-mail address: Anna.Nazarova@uni-due.de