A new approach to storage value modelling

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Abstract. We suggest a simple and useful approach to compute the value of a hydro- and gas-driven storage facilities. Instead of implementing a common stochastic control method, we assume that we are already given with an optimal policy. With this at hand we suggest various payoffs that help a producer to hedge the market position and to compute its value.

Keywords: Storage value process, diffusion in a target zone, energy markets

1. Introduction and literature

The problem of modelling storage is not new, but still very demanding and challenging due to its direct purpose of matching the supply and demand in energy markets. The key purposes of controlling a storage process include keeping the balance in the reservoir, meeting the changing demand, hedging market positions, insuring against various sudden events, performing market speculations and others.

To meet the seasonal changes in demand both renewable- and fuel-driven production industries have storage reservoirs. For example, the hydro-dominated Nordic power market (i.e. Norway and Sweden) represents private and public parties operating hydro reservoirs. As stated in Kauppi and Liski (2008), this hydro system has some specific features such as weather-dependency in spring and fall, many different inflow and outflow technical constraints in hydro turbines and other. Since inflow is highly seasonal and can show some instability, there is a strong interconnection between the Scandinavian markets. For instance, depending on the conditions, the necessary amount of hydro power can safely be transported from one region to another. Moreover, there exists a cumulative hydro storage index, available at Nord Pool exchange, that shows current hydro reservoir levels across countries and in the total maximum capacity. (**add some more descriptive issues on the hydro storage here**).

Other storage alternatives to have a quick access to are facilities to easily store fuels, mainly gas. These facilities have some specific properties and characteristics one should keep in mind. Among them are the reservoir capacity constraint and injection and withdrawal rate constraints. The latter rate constraint regulates the speed of injection or withdrawal depending on the current reservoir level. Other important operating characteristics are the base gas (cushion) level that ensures the critical pressure in the pipeline and the working gas level which allows one to operate in the market. Also, there is a cyclability constraint

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representing a number of cycles of injection or withdrawal per year. Furthermore gas storage entails various operational and managerial costs. Additional to these costs there are possible pipeline seepage rates which describe the amount of gas that is lost during injection or withdrawal. On top of that there might be some regulatory constraints.

Technically, there exist three types of underground gas storage facilities: salt caverns, aquifers and depleted oil or gas reservoirs, see Commission (2004). The first type of facility has relatively high deliverability and injection rates and is often used for short-term purposes. The second type of facility has high cushion level requirements and a high deliverability rate. The last one is the most common gas storage provision and is used for seasonal system supply or for peak-day demands.

As the reader can see, hydro storage and gas storage problems have some issues in common. Particularly, the hydro storage problem addresses the questions of when and how much water to release or to save and how much power to produce respectively. The gas storage problem addresses the questions of when to withdraw and sell and when to buy in the market and inject. While the former problem has not extensively been discussed in the literature (**other papers to mention here?**), the latter problem was under a quite focus for the last decade in the literature. Papers of Ahn, Danilova, and Swindle (2002), Chen and Forsyth (2007), Kjaer and Ronn (2008), Thompson, Davison, and Rasmussen (2009), Carmona and Ludkovski (2010) investigate the working gas storage value problem as a stochastic control problem.

Namely, they consider the control policy which defines the periods of injection, withdrawal or "doing nothing" in such a way that the total profit of a storage holder is maximised with respect to some constraints. More precisely, there is a physical storage at level S_t which is limited up to the maximal storage capacity S_{max}. The market price of gas P_t can either be considered as a futures price $F(t,t)$ with some respective assumptions on $F(t,T)$ or can alternatively be modelled as a stochastic mean-reverting process possibly with jumps. There are two considered rates: injection $a_{in}(S_t) > 0$ and withdrawal $a_{out}(S_t) < 0$, not necessarily equal to each other by their absolute value. There are possibly some costs of injection/withdrawal together with some other operational and managerial costs of storage. Furthermore there is a finite (or infinite) horizon with either continuous (or discrete) time setting. All this sets up the following optimisation problem of finding an optimal switching policy between injection, withdrawal or "doing nothing" regimes. This problem belongs to a class of stochastic control problems, since one seeks for an optimal strategy c from the class $\mathcal{C}_t$ of all admissible strategies. Given starting values at time t one has the following formulation

$$V(t, P_t, S_t) = \sup_{c \in \mathcal{C}_t} \mathbb{E}\Big[\int_t^T h(c_s, P_s, S_s)\,\mathrm{d}s\Big], \tag{1}$$

where h is a specified payoff that an investor receives at time t implementing the strategy c. Depending on the assumptions and modelling properties, this Hamilton-Jacobi-Bellman type of problem can be solved with several techniques extensively available in the literature. The resulting optimal stochastic control policy explains three regimes: if the working gas level in storage is low, then with the gas price increasing one is moving from a strategy of pumping up to "doing nothing" with gas. Conversely, if the working gas level in the storage facility is high, then with the gas price increasing one has an opportunity of releasing gas from storage to sell. This control strategy corresponds to the following policy: one sells the gas and it results in the highest value when the prices are high and the reservoir is full. Respectively, if the prices are high and the reservoir is empty, then one neither sells or buys. In this paper we will not focus on solving the stochastic optimal control problem, instead we assume that the optimal policy of injection or withdrawal is given and investigate a number of important financial products that a producer can use in order to hedge the market position.

2. Motivation

In a contrast to the approach described above, we look at the problem differently and develop a method that allows us to analyse both the gas and hydro storage problems from the stochastic modelling and statistical points of view. We consider a working storage (gas or hydro) as a mean-reverting bounded stochastic process, implying that the control policy to inject or withdraw is already given.

The motivation for modelling storage level as a random process is the following. Consider a producer who owns a storage reservoir: she has to regularly decide on injection/withdrawal/"doing nothing" action depending on various external factors. When she deals with the hydro storage problem, one of the key factors is the power price P_t. More precisely, the producer tends to release the water to produce power if the current power price level is relatively high and if the current water level in the reservoir allows one to do so. Alternatively, if the current price level is low, one can only opt for a base production. When the operator deals with the gas storage problem, then one of these factors has been considered in the literature as a gas spot price, also P_t. [2] However, since the spot price contains information up to time t, we would suggest that this producer looks at the futures market and takes a decision respectively. If the market is currently in contango, meaning that the value $D_t = F(t,T) - \mathbb{E}^{\mathbb{Q}}[P_T \mid \mathcal{F}_t] > 0$, then the producer can expect that the market is willing to pay more in the future. The opposite case is the backwardation, meaning that the value $D_t = F(t,T) - \mathbb{E}^{\mathbb{Q}}[P_T \mid \mathcal{F}_t] < 0$, then the producer can expect that the prices will be lower in the future. So this would help to either inject during contango or withdraw during backwardation. Since one usually observes contango in summer and backwardation in winter, we can think of the value D_t as a process which is reverting around zero. This would imply that if we follow the strategy to inject when the market is in contango the storage level is below some mean level m. If we follow the strategy to withdraw when the market is in backwardation the storage level is above the mean level m.

Since demand is highly seasonal, managing inventories plays a big role in various risk hedging methods. A stochastic model for storage which does not include the stochastic control component would shed some light on the storage value dynamics and gives an intuition to hedging against a price collapse or other unexpected events. Another benefit of such a setup is that it gives a quick and simple way to estimate the value of owning a storage facility knowing the current market price.

[2] Further in the text we refer P_t as a fuel price which can either be power or gas price respectively.

The paper is structured as follows. Chapter 3 introduces the model with all the necessary components for storage dynamics, spot market price dynamics and the value process. For the sake of comparison, we also consider several payoffs for hydro and gas storage problems respectively. Chapter 4 gives several illustrative examples. Chapter 5 provides a discussion and chapter 6 concludes.

3. Storage Process Modelling

3.1. **Modelling setup.** Let $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\})$ be a complete filtered probability space. We specify the model assumptions and parameters:

- continuous time setting;
- finite time horizon $t, T \in [T_1, T_2]$;
- S_t is the current level of working hydro/gas in the reservoir at moment t measured in MWh;
- reservoir capacity is restricted naturally by $0 < l < u < +\infty$ with l is the minimum reservoir level and u the maximum reservoir level;
- $a(S_t)$ is the rate at which we inject or withdraw;
- P_t is the spot price (gas or power);
- $F(t,T)$ is the futures price (gas or power) with maturity T;
- $V_t(S_t, P_t, C_t)$ is the storage value at time t;
- $r(t,T)$ is the discount factor over the period of (t,T);
- there are some cumulative (operational, managerial or switching) costs C_t included.

We model the storage level dynamics S_t as a stochastic mean-reverting process which stays between (l,u) as follows

$$\mathrm{d}S_t = -2(S_t - m)\ \mathrm{d}t + \sqrt{2(S_t - l)(u - S_t)}\ \mathrm{d}W_t^S, \tag{2}$$

where $m = \frac{u+l}{2}$ is the average reservoir level. An illustrative example of such a process is given in Figure 1. This formulation suggests that the injection and withdrawal rates are defined by $\mathrm{d}S_t$. The drift term becomes positive when the reservoir is relatively empty and needs to be re-filled and the drift term becomes negative when the reservoir is relatively full and needs to be emptied. The diffusion term ensures the fact that the process S_t always stays inside the interval (l,u) and never reaches the boundaries l and u, which is exactly the case for the real storage level process due to regulatory constraints on the minimum and maximum reservoir levels l and u.

In this paper we will focus on the hydro- and gas-driven storage reservoirs. The hydro-driven reservoir is naturally filled with melted snow or rain. So the amount of precipitation can be regarded as a random process. In (REFERENCE to Stein Erik Fleten paper) the authors provide data from Norwegian producers operating hydro storage reservoirs. Their data show the random nature of the inflow process. Another important issue discussed in the paper is the so-called target level set by the producer. This target level is time-dependent that can be explained by the seasonal behavior of the inflow to the storage facility: due to high power demand in winter and low power demand in summer. This effect can be captured by incorporating some circular (e.g. trigonometric) function. In our model for the sake of tractability and simplicity we refer to this seasonal component as constant level m, namely we assume that $m := \mathbb{E}[m(t)]$.

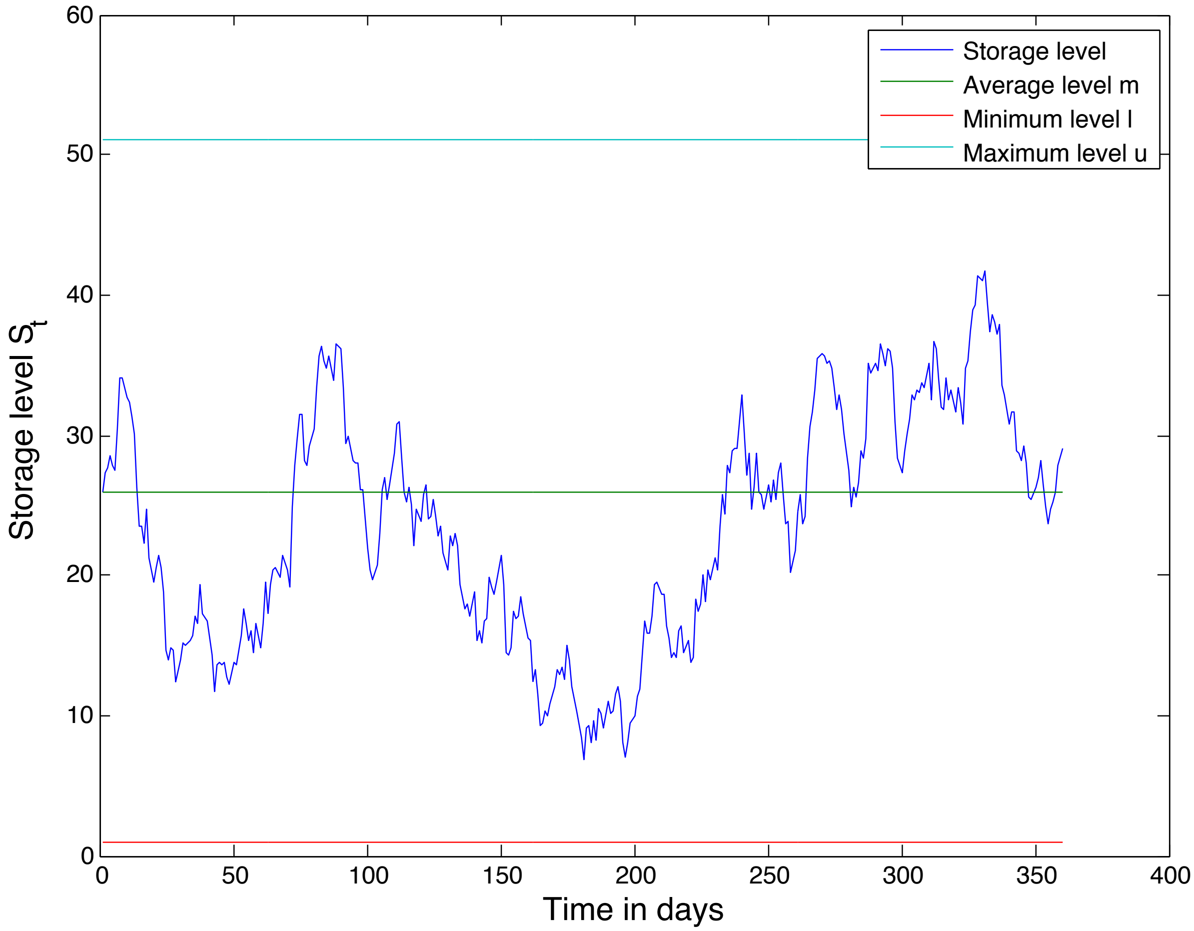


FIGURE 1. An example of a storage level process with $l = 1$ and $u = 51$.

Need to mention the diffusion in a target zone and papers of de jong and sorensen and their application to a currency exchange markets.

The spot price dynamics is described by an exponential Ornstein-Uhlenbeck process without jumps which ensures the price positivity (here we ignore the fact that sometimes power can exhibit negative prices), namely

$$\begin{aligned} P_t &= e^{f(t)+X_t}, \\ \mathrm{d}X_t &= -\alpha X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t^X, \\ \mathrm{d}P_t &= \alpha\Big(\mu(t) - \log P_t\Big) P_t \,\mathrm{d}t + \sigma P_t \,\mathrm{d}W_t^X, \end{aligned} \tag{3}$$

where α is the speed of mean-reversion to the mean level $f(t)$ (possibly capturing the seasonal component), σ is a constant volatility and $\mu(t) := \frac{1}{\alpha}(\frac{\sigma^2}{2} + f'_t) + f(t)$.

We also assume the following correlation structure with

$$\mathrm{d}W_t^X\,\mathrm{d}W_t^S = \rho\,\mathrm{d}t. \tag{4}$$

Then we continue with assuming linear consistency on the correlation structure: if $\rho = \mathrm{corr}(X_t, S_t) < 0$, then $\hat{\rho} = \mathrm{corr}\big((X_t - x)\mathbb{1}_{X_t > x}, (S_t - s)\mathbb{1}_{S_t > s}\big) < 0$. To motivate this assumption, one can think of the following: if the storage facilities are relatively full or increasing (e.g., extra precipitation) and market is aware of the lack of a storable asset shortage, then the market power price would be relatively low or respectively decreasing.

Now we can introduce the value process $V_t(P_t, S_t, C_t)$ as

$$V_t(P_t, S_t, C_t) = \mathbb{E}\Big[\int_t^T e^{-r(s,t)}\, H_s(P_s, S_s, C_s)\,\mathrm{d}s|\mathcal{F}_t\Big], \tag{5}$$

where $H_t(P_t, S_t, C_t)$ is a payoff including the various costs C_t and $r(t,T)$ is a discount factor. Simply speaking, we consider a value process as a discounted payoff which is a combination of two stochastic processes (power or gas price and storage level). Since we know the statistical properties of these two processes, we aim to investigate their product process to have some approximation of on the storage value process.

Before we proceed with the investigation of various payoffs, we need to recall some technical properties of the processes P_t and S_t. It is known fact from the Equation (3) that the logarithm of the price P_t is Gaussian. So one can explicitly compute the mean and the variance of both $\log P_t$ and P_t. Regarding the process S_t, we can characterise it by the transition density function $p_{t-t_0}(x,y)$. We will use the following notations: $\Delta := u - l$, $c_n = \frac{\Delta}{2}$, $v_n(x) := \sqrt{\frac{2n+1}{2}} P_n\Big(\frac{2}{\Delta}(x-l) - 1\Big)$ with $P_n(x)$ being Legendre's[3] series of order n and $C(n,x,t) := \frac{2n+1}{\Delta} P_n\Big(\frac{2}{\Delta}(x-l) - 1\Big) e^{-n(n+1)(t-t_0)}$. Then the transition density function $p_{t-t_0}(x,y)$ derived in REFERENCE is given as

$$\begin{aligned} p_{t-t_0}(x,y)\,|_{(l,u)} &= \sum_{n=0}^{\infty} \frac{v_n(x)v_n(y)}{c_n}\, e^{-n(n+1)(t-t_0)} \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta}\, P_n\Big(\frac{2}{\Delta}(x-l) - 1\Big) P_n\Big(\frac{2}{\Delta}(y-l) - 1\Big)\, e^{-n(n+1)(t-t_0)} \\ &= \sum_{n=0}^{\infty} C(n,x,t) P_n\Big(\frac{2}{\Delta}(y-l) - 1\Big). \end{aligned} \tag{6}$$

[3]There is a variety of literature on the Legendre polynomials available, for instance, Whittaker and Watson (1996) and Bell (2004). The first few polynomials are: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, $P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$.

We will also use the following properties of the Legendre series (for the details see Abramowitz and Stegun (1970), page 786 and Bell (2004), pages 56-58):

- if $f(z)$ is a polynomial with a degree less than $P_n(z)$ then

$$\int_{-1}^{1} f(z)P_n(z)\ \mathrm{d}z = 0; \tag{7}$$

- for $n \geq 1$

$$\int_{x}^{1} P_n(z)\ \mathrm{d}z = \frac{P_{n-1}(x) - P_{n+1}(x)}{2n+1}, \tag{8}$$

  we denote this quantity as $P^*_{n-1,n+1}(x)$ for future calculations;
- for $n \geq 2$

$$\int_{x}^{1} zP_n(z)\ \mathrm{d}z = \frac{n(2n+3)P_{n-2}(x) - (2n+1)P_n(x) - (n+1)(2n-1)P_{n+2}(x)}{(4n^2-1)(2n+3)}, \tag{9}$$

  we denote this quantity as $P^*_{n-2,n,n+2}(x)$ for future calculations;
- for $n \geq 3$

$$\begin{aligned} \int_{x}^{1} z^2P_n(z)\ \mathrm{d}z &= \frac{n(n-1)}{(4n^2-1)(2n-3)}P_{n-3}(x) - \frac{(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)}P_{n+3}(x) \\ &- \frac{n^2+3n-1}{(4n^2-1)(2n+5)}P_{n+1}(x) + \frac{n^2-n-3}{(4n^2-9)(2n+1)}P_{n-1}(x), \end{aligned} \tag{10}$$

  we denote this quantity as $P^*_{n-3,n+3}(x)$ for future calculations;
- for even n

$$\int_{0}^{1} z^2P_n(z)\ \mathrm{d}z = \frac{(-1)^n(n-1)(3/2)}{2(-1)(n+5/2)}, \tag{11}$$

  we denote this quantity as P^*_{2n} for future calculations;
- for odd n

$$\int_{0}^{1} z^2P_n(z)\ \mathrm{d}z = \frac{(-1)^n(n-1/2)(2)}{2(n+3)(-1/2)}, \tag{12}$$

  we denote this quantity as P^*_{2n+1} for future calculations;

We will use the following expression and notation for the expected value of the process $(S_t - m)$

$$
\begin{aligned}
E_0(t,T) &:= \mathbb{E}\big[S_T - m|\mathcal{F}_t\big] \\
&= \int_l^u (S_T - m)\, p_{T-t}(S_t, S_T)\, \mathrm{d}S_T \\
&= \int_l^u (y - m)\, p_{T-t}(x, y)\, \mathrm{d}y \\
&= \int_l^u (y-m) \Big(\sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n \left(\frac{2}{\Delta}(y-l) - 1 \right) P_n \left(\frac{2}{\Delta}(x-l) - 1 \right) e^{-n(n+1)(T-t)} \Big)\, \mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n \left(\frac{2}{\Delta}(x-l) - 1 \right) e^{-n(n+1)(T-t)} \int_l^u (y-m) P_n \left(\frac{2}{\Delta}(y-l) - 1 \right)\, \mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n \left(\frac{2}{\Delta}(x-l) - 1 \right) e^{-n(n+1)(T-t)} \frac{\Delta^2}{4} \int_{-1}^{1} z P_n\big(z\big)\, \mathrm{d}z \\
&= \frac{\Delta}{4} \int_{-1}^{1} z\, \mathrm{d}z + \frac{3\Delta}{4} P_1 \left(\frac{2}{\Delta}(x-l) - 1 \right) e^{-2(T-t)} \int_{-1}^{1} z^2\, \mathrm{d}z \\
&+ \underbrace{\sum_{n=2}^{\infty} \frac{2n+1}{\Delta} P_n \left(\frac{2}{\Delta}(x-l) - 1 \right) e^{-n(n+1)(T-t)} \frac{\Delta^2}{4} \int_{-1}^{1} z\, P_n(z)\, \mathrm{d}z}_{=0,\,\text{due to Equation (7)}} \\
&= (S_t - m) e^{-2(T-t)}, \qquad (13)
\end{aligned}
$$

where we use a substitution $z := \frac{2}{\Delta}(y-l) - 1$.

We will use the following expression and notation for the variance of the process $(S_t - m)$ (which is of course the same as variance of S_t but it's useful to have this notation for a future purposes)

$$
\begin{aligned}
V_0(t,T) &:= \mathrm{Var}\big[S_T - m|\mathcal{F}_t\big] \\
&= \mathbb{E}\big[(S_T - m)^2|\mathcal{F}_t\big] - \Big(\mathbb{E}\big[S_T - m|\mathcal{F}_t\big]\Big)^2 \\
&= \int_l^u (S_T - m)^2\, p_{T-t}(S_t, S_T)\, \mathrm{d}S_T - (S_t - m)^2 e^{-4(T-t)} \\
&= \int_l^u (y - m)^2\, p_{T-t}(x, y)\, \mathrm{d}y - (S_t - m)^2 e^{-4(T-t)} \\
&= \int_l^u (y-m)^2 \Big(\sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n \left(\frac{2}{\Delta}(y-l) - 1 \right) P_n \left(\frac{2}{\Delta}(x-l) - 1 \right) e^{-n(n+1)(T-t)} \Big)\, \mathrm{d}y \\
&- (S_t - m)^2 e^{-4(T-t)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(T-t)} \frac{\Delta^3}{8} \int_{-1}^{1} z^2 P_n(z)\,\mathrm{d}z \\
&= \frac{\Delta^2}{8} \int_{-1}^{1} z^2\,\mathrm{d}z + \frac{3\Delta^2}{8} P_1\left(\frac{2}{\Delta}(x-l)-1\right) e^{-2(T-t)} \int_{-1}^{1} z^3\,\mathrm{d}z \\
&+ \frac{5\Delta^2}{32} P_2\left(\frac{2}{\Delta}(x-l)-1\right) e^{-6(T-t)} \int_{-1}^{1} z^2 P_2(z)\,\mathrm{d}z \\
&+ \underbrace{\sum_{n=3}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(T-t)} \frac{\Delta^3}{8} \int_{-1}^{1} z^2\, P_n(z)\,\mathrm{d}z}_{=0,\ \text{due to Equation (7)}} \\
&- (S_t-m)^2 e^{-4(T-t)} \\
&= \frac{\Delta^2}{12} + e^{-6(T-t)}\left((S_t-m)^2 - \frac{\Delta^2}{12}\right) - (S_t-m)^2 e^{-4(T-t)} \\
(14) &= (S_t-m)^2 e^{-4(T-t)}\left(e^{-2(T-t)}-1\right) + \frac{\Delta^2}{12}\left(1-e^{-6(T-t)}\right),
\end{aligned}
$$

where we also substitute $z := \frac{2}{\Delta}(y-l)-1$. We will use the following expression and notation for the expected value of P_t

$$
E_1(t,T) := \mathbb{E}\big[P_T|\mathcal{F}_t\big] = \exp\left(f(T) + X_t e^{-\alpha(T-t)} + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)})\right). \tag{15}
$$

We will use the following expression and notation for the variance of the process P_t

$$
\begin{aligned}
V_1(t,T) &:= \mathrm{Var}\big[P_T|\mathcal{F}_t\big] \\
&= \Big(\mathbb{E}\big[P_T|\mathcal{F}_t\big]\Big)^2\left(\exp\left(\frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(T-t)}\right)\right)-1\right) \\
&= E_1^2(t,T)\left(e^{\frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(T-t)}\right)}-1\right).
\end{aligned} \tag{16}
$$

In the next sections we start considering simple products such as forwards and options on the storage level. Subsequently, we continue with various payoffs linking the power price and the storage level processes together.

3.2. **Probability measure.** Before we start with pricing, we need to clarify some points on a pricing measure. From mathematical finance theory we know that in a complete market a contingent claim's price is the discounted expected value of the future payoff under the **equivalent martingale measure** $\mathbb{Q}$ different from a real-world pricing measure $\mathbb{P}$. However, the energy-related markets are incomplete, since due to specific market characteristics many payoffs cannot be replicated by other trading financial instruments. In our case the "spot price" is the storage level process which can be, for instance, considered as an index of current state reservoir level (hydro). Hence, we cannot think of $\mathbb{Q}$ being the martingale measure since the process S_t does not need to be a martingale under $\mathbb{Q}$. Instead, we can take **any** measure $\mathbb{Q}$ equivalent to the real-world measure $\mathbb{P}$, i.e. $\mathbb{Q} = \mathbb{P}$, and price derivatives respectively. So then this measure can be called as a **pricing** measure which is the probability measure that takes into account all the risk associated with maintaing the storage. In other words, we suppose that the process S_t is already under pricing measure $\mathbb{Q} = \mathbb{P}$.

3.3. **Simple financial products.** Equipped with the Legendre series properties together with the expression for the transition probability density $p_{t-t_0}(x, y)$ in Equation (6), we now study some fundamental financial products: futures and options on a reservoir level. Since the storage level S_t at time t is a random process, these financial instruments gamble that the current reservoir level S_t rise or fall above or below some level. They can also be used by the producer, retailer or market maker to hedge their risk when maintaining the storage or, for example, to speculate (since the power price can be quite volatile). The risks are various. Consider, for instance, the case when the power price is high and the water level is low. Then our producer and/or storage owner does not have a chance to produce and has a potential loss. Another case is when the power price is low and the water level is high, the producer and/or storage owner still bears the costs on maintaining the storage, but it is not profitable to produce power due to low power price level. Since the high power price volatility is a constant source of uncertainty and risk, the producer is willing to hedge against it, especially if she has the fixed price contracts.

3.3.1. *Forward on the reservoir level.* Under some pricing measure $\mathbb{Q} = \mathbb{P}$ we can due to Equation (13) write the futures price on the water (or gas) level with maturity T as

$$\begin{aligned} F(t,T) &= \mathbb{E}[S_T|\mathcal{F}_t] \\ &= S_t e^{-2(T-t)} + m(1 - e^{-2(T-t)}). \end{aligned} \tag{17}$$

We can also price futures on the average water level deviation over some period of time by considering

$$\begin{aligned} F(t,T_1,T_2) &= \frac{1}{T_2 - T_1}\,\mathbb{E}\Big[\int_{T_1}^{T_2} (S_u - m)\,\mathrm{d}u|\mathcal{F}_t\Big] \\ &= \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathbb{E}\Big[S_u - m|\mathcal{F}_t\Big]\,\mathrm{d}u \\ &= \frac{1}{T_2 - T_1}\,\frac{1}{2}\,(S_t - m)\big(e^{-2(T_1-t)} - e^{-2(T_2-t)}\big). \end{aligned} \tag{18}$$

3.3.2. *European options on the reservoir level.* Call or Put on the water (or gas) level in the reservoir.

Let us start with a European call option and some strike K which can be interpreted as marginal cost for maintaining the reservoir (re-check validity of second equality)

$$C(t,T) = \mathbb{E}^Q[\max\{S_T - K, 0\}|\mathcal{F}_t]$$

$$
\begin{aligned}
&= \mathbb{E}^Q[\max\{S_T - K, 0\}|S_t] \\
&= \int_K^u (y-K) p_{T-t}(x,y)\,\mathrm{d}y \\
&= \int_K^u (y-K) p_{T-t}(x,y)\,\mathrm{d}y \\
&= \int_K^u (y-K)\Big(\sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(y-l)-1\right) P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(T-t)}\Big)\,\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(T-t)} \int_K^u (y-K) P_n\left(\frac{2}{\Delta}(y-l)-1\right)\,\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(T-t)} \int_{\tilde{K}}^1 \frac{\Delta}{2}\Big(\frac{\Delta}{2}z + (m-K)\Big) P_n(z)\,\mathrm{d}z \\
&= \frac{1}{\Delta}\Big(\frac{\Delta^2}{4}\frac{1-\tilde{K}^2}{2} + \frac{\Delta}{2}(m-K)(1-\tilde{K})\Big) \\
&+ \frac{3}{\Delta} P_1\left(\frac{2}{\Delta}(S_t-l)-1\right) e^{-2(T-t)}\Big(\frac{\Delta^2}{4}\frac{1-\tilde{K}^3}{3} + \frac{\Delta}{2}(m-K)\frac{1-\tilde{K}^2}{2}\Big) \\
&+ \sum_{n=2}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(S_t-l)-1\right) e^{-n(n+1)(T-t)}\Big(\frac{\Delta^2}{4} P^*_{n-2,n,n+2}(\tilde{K}) + \frac{\Delta}{2}(m-K) P^*_{n-1,n+1}(\tilde{K})\Big),
\end{aligned}
\tag{19}
$$

where we made the replacements $z := \frac{2}{\Delta}(y-l)-1$ and $\tilde{K} := \frac{2}{\Delta}(K-l)-1$. Expressions for $P^*_{n-1,n+1}(x)$ and $P^*_{n-2,n,n+2}(x)$ are given above in Equations (8) and (9) respectively.

3.4. **Hydro-Driven Power Plant.** In this section we consider payoffs which can be used to construct the value of a hydro-driven power plant and study its properties in a similar manner as in Nazarova, Kiesel, Bannör, and Scherer (2013). In general, we consider a producer who wants to hedge against some unfavorable situations like too low a water level in the reservoir, and too low or high prices. Therefore, such a producer could be interested in an option with which she can hedge against both water levels and price, as low water does not necessarily lead to high prices, only if demand is very high at the same time. Our producer might have contracts that she needs to fulfill with fixed prices, and thereby is concerned with too high a price or too low a water level. But too low a water level and too low a price may be connected with above average temperatures, and then the producer does not risk that much since she does not need to retail much power anyway. Since simple financial products only consider the current level in the storage facility, it is not enough to hedge against various complex cases, for these one needs to have advanced financial products with complicated payoffs.

3.4.1. *Payoff 1.* Consider a hydro-driven power plant and a payoff (its form is quite similar to quanto options studied in Benth, Lange, and Myklebust (2013)) that includes an average power price level M and average storage level m, namely

$$H_t(P_t, S_t, C_t) = \max\{P_t - M, 0\} \times \max\{S_t - m, 0\} - C_t. \tag{20}$$

Since the hydro reservoir depends on the natural inflow and we cannot "inject" any water, this payoff has the following interpretation:

- **Case 1: $P_t > M$ (power prices are relatively high) and $S_t > m$ (reservoir is full).** This is the most favourable situation which results in a positive value that investor can have by releasing some water (difference between current S_t and average level m), producing power and selling it at the market.
- **Case 2: $P_t > M$ (power prices are relatively high) and $S_t < m$ (reservoir is relatively empty).** This situation corresponds to the "doing nothing" regime, i.e. we don't have much water in the reservoir to produce power, though we may like to keep the production routine.

Being at time t we find that

$$\begin{aligned}
V_t(P_t, S_t) &= \mathbb{E}\Big[\int_t^T e^{-r(k,t)}\, H_k(P_k, S_k, C_k)\,\mathrm{d}k|\mathcal{F}_t\Big] \\
&= \int_t^T e^{-r(k,t)} \Big(\mathbb{E}\big[H_k(P_k, S_k)|\mathcal{F}_t\big]\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)} \Big(\mathbb{E}\big[\max\{P_k - M, 0\} \times \max\{S_k - m, 0\}|\mathcal{F}_t\big] - C_k\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)} \Big(\underbrace{\mathbb{E}\big[\max\{P_k - M, 0\}|\mathcal{F}_t\big]}_{=:E_3(t,k)}\underbrace{\mathbb{E}\big[\max\{S_k - m, 0\}|\mathcal{F}_t\big]}_{=:E_2(t,k)} \\
&\quad + \rho_1 \sqrt{\underbrace{\mathrm{Var}(\max\{P_k - M, 0\}|\mathcal{F}_t)}_{=:V_3(t,k)}\underbrace{\mathrm{Var}(\max\{S_k - m, 0\}|\mathcal{F}_t)}_{=:V_2(t,k)}} - C_k\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)} \Big(E_3(t,k)\cdot E_2(t,k) + \rho_1 \cdot \sqrt{V_3(t,k)\cdot V_2(t,k)} - C_k\Big)\,\mathrm{d}k, \qquad (21)
\end{aligned}$$

where $\rho_1 = \mathrm{corr}(\max\{P_k - M, 0\}, \max\{S_k - m, 0\})$.

Let us now compute the values E_2, E_3, V_2 and V_3. From Equation (3) the log price is Gaussian, i.e. for $k > t$ we have that $\ln P_k = (f(k) + X_k) \sim \mathcal{N}(\underbrace{f(k) + X_t e^{-\alpha(k-t)}}_{=:m_1}, \underbrace{\sigma^2(1 - e^{-2\alpha(k-t)})/(2\alpha)}_{=:m_2})$.

So, we obtain

$$\begin{aligned}
E_3(t,k) &:= \mathbb{E}\big[\max\{P_k - M, 0\}|\mathcal{F}_t\big] \\
&= \mathbb{E}\big[\max\{e^{f(k)+X_k} - M, 0\}|\mathcal{F}_t\big] \\
&= \int_{\ln M}^{\infty} (e^y - M)\phi(y, x, t)\ \mathrm{d}y \\
&= e^{m_1 + \frac{m_2}{2}}\Phi\Big(\frac{m_1 + m_2 - \ln M}{\sqrt{m_2}}\Big) - M\Phi\Big(\frac{m_1 - \ln M}{\sqrt{m_2}}\Big) \\
&= e^{m_1 + \frac{m_2}{2}}\Phi(d_2) - M\Phi(d_1),
\end{aligned}$$

where $d_1 := \frac{f(k)+X_t e^{-\alpha(k-t)} - \ln M}{\sqrt{\sigma^2/(2\alpha)(1-e^{-2\alpha(k-t)})}}$ and $d_2 := d_1 + \sqrt{\sigma^2/(2\alpha)(1-e^{-2\alpha(k-t)})}$.

Further, knowing the transition density function $p_{t-t_0}(x,y)$ for the process $S_t \in (l,u)$, we obtain

$$
\begin{aligned}
E_2(t,k) \;&:=\; \mathbb{E}\big[\max\{S_k - m, 0\} | \mathcal{F}_t\big] \\
&= \int_m^u (y-m)\, p_{k-t}(x,y)\ \mathrm{d}y \\
&= \int_m^u (y-m) \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) P_n\left(\frac{2}{\Delta}(y-l)-1\right) e^{-n(n+1)(k-t)}\ \mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)} \int_m^u (y-m) P_n\left(\frac{2}{\Delta}(y-l)-1\right)\ \mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)} \frac{\Delta^2}{4} \int_0^1 z P_n(z)\ \mathrm{d}z \\
&= \frac{\Delta}{8} + \frac{\Delta}{4} P_1\left(\frac{2}{\Delta}(x-l)-1\right) e^{-2(k-t)} \\
&+ \sum_{n=2}^{\infty} \frac{(2n+1)\Delta}{4} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)}\, P^*_{n-2,n,n+2}(0) \\
&= \frac{\Delta}{8} + \frac{\Delta}{4} P_1\left(\frac{2}{\Delta}(S_t-l)-1\right) e^{-2(k-t)} \\
&+ \sum_{n=2}^{\infty} \frac{(2n+1)\Delta}{4} P_n\left(\frac{2}{\Delta}(S_t-l)-1\right) e^{-n(n+1)(k-t)}\, P^*_{n-2,n,n+2}(0), \qquad (22)
\end{aligned}
$$

where the expression for $P^*_{n-2,n,n+2}(0)$ is given in Equation (9). Now let us compute value V_3

$$
\begin{aligned}
V_3(t,k) \;&:=\; \mathrm{Var}(\max\{P_k - M, 0\} | \mathcal{F}_t) \\
&= \mathrm{Var}(\max\{e^{f(k)+X_k} - M, 0\} | \mathcal{F}_t) \\
&= \int_{\ln M}^{\infty} (e^y - M)^2 \phi(y,x,t)\ \mathrm{d}y - \Big(\int_{\ln M}^{\infty} (e^y - M)\phi(y,x,t)\ \mathrm{d}y\Big)^2 \\
&= \int_{\ln M}^{\infty} e^{2y} \phi(y,x,t)\ \mathrm{d}y \\
&+ M\big(1 - \int_{\ln M}^{\infty} \phi(y,x,t)\ \mathrm{d}y\big)\big(M \int_{\ln M}^{\infty} \phi(y,x,t)\ \mathrm{d}y - 2\int_{\ln M}^{\infty} e^y \phi(y,x,t)\ \mathrm{d}y\big) \\
&- \big(\int_{\ln M}^{\infty} e^y \phi(y,x,t)\ \mathrm{d}y\big)^2 \\
&= e^{2(m_1+m_2)} \Phi\Big(\frac{m_1 + 2m_2 - \ln M}{\sqrt{m_2}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad M\Phi\Big(\frac{\ln M - m_1}{\sqrt{m_2}}\Big)\Big(M\Phi\Big(\frac{m_1 - \ln M}{\sqrt{m_2}}\Big) - 2e^{m_1+\frac{m_2}{2}}\Phi\Big(\frac{m_1+m_2-\ln M}{\sqrt{m_2}}\Big)\Big) \\
&- \quad e^{2m_1+m_2}\Phi^2\Big(\frac{m_1+m_2-\ln M}{\sqrt{m_2}}\Big) \\
&= \quad e^{2(m_1+m_2)}\Phi(d_3) + M\Phi(-d_1)\Big(M\Phi(d_1) - 2e^{m_1+\frac{m_2}{2}}\Phi(d_2)\Big) - e^{2m_1+m_2}\Phi^2(d_2),
\end{aligned} \tag{23}
$$

where d_1 and d_2 are given above and $d_3 := d_1 + 2\sqrt{\sigma^2/(2\alpha)(1-e^{-2\alpha(k-t)})}$.

And finally we compute value V_2

$$
\begin{aligned}
V_2(t,k) &:= \operatorname{Var}(\max\{S_k - m, 0\}|\mathcal{F}_t) \\
&= \int_m^u (y-m)^2 p_{t-k}(x,y)\,\mathrm{d}y - E_2^2(t,k) \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)} \int_m^u (y-m)^2 P_n\left(\frac{2}{\Delta}(y-l)-1\right)\,\mathrm{d}y - E_2^2(t,k) \\
&= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)} \frac{\Delta^3}{8}\int_0^1 z^2 P_n(z)\,\mathrm{d}z - E_2^2(t,k), \\
&= \sum_{n=0}^{\infty} \frac{(2n+1)\Delta^2}{8} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)} \{P_{2n}^*, P_{2n+1}^*\} - E_2^2(t,k),
\end{aligned} \tag{24}
$$

where P_{2n}^* and P_{2n+1}^* are given in Equations (11) and (12) respectively.

3.4.2. *Payoff 2.* Let us the modify the payoff above and introduce a payoff function that has an extra term responsible for the power production rate

$$
\kappa(S_t) = \frac{1}{2} + \frac{S_t - m}{u-l} = \frac{S_t - l}{u-l},
$$

$$
H_t(P_t, S_t, C_t) = \max\{P_t - M, 0\} \times \kappa(S_t) \times S_t - C_t. \tag{25}
$$

The difference to the previous payoff is that in this case we can produce at the rate κ which is greater than 50% if the $S_t > m$. There is the following interpretation for this payoff allowing for more flexibility in the production rate compared to the previous one:

- **Case 1: $P_t > M$ (power prices are relatively high) and $S_t > m$ (reservoir is full).** This is the most favourable situation which results in a positive value that investor has by releasing some water, producing power at the rate $\kappa(S_t)$ and selling it at the market.
- **Case 2: $P_t > M$ (power prices are relatively high) and $S_t < m$ (reservoir is relatively empty).** This situation corresponds to the base regime, i.e. we don't have much water in the reservoir to produce intensively, but since the prices are high we wouldn't like to completely stop operating.

Being at time t we find that

$$
\begin{aligned}
V_t(P_t, S_t, C_t) &= \mathbb{E}\Big[\int_t^T e^{-r(k,t)}\, H_k(P_k, S_k, C_t)\,\mathrm{d}k|\mathcal{F}_t\Big] \\
&= \int_t^T e^{-r(k,t)}\,\Big(\mathbb{E}\big[H_k(P_k,S_k)|\mathcal{F}_t\big]\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\,\Big(\mathbb{E}\big[\max\{P_k - M, 0\} \times \kappa(S_k) \times S_k|\mathcal{F}_t\big] - C_k\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\,\Big(\mathbb{E}\Big[\max\{P_k - M, 0\} \times \frac{S_k(S_k - l)}{u-l}|\mathcal{F}_t\Big] - C_k\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\,\Big(\underbrace{\mathbb{E}\big[\max\{P_k - M, 0\}|\mathcal{F}_t\big]}_{=:E_3(t,k)}\underbrace{\mathbb{E}\Big[\frac{S_k(S_k-l)}{u-l}|\mathcal{F}_t\Big]}_{=:E_4(t,k)} \\
&+ \rho_2\sqrt{\underbrace{\mathrm{Var}(\max\{P_k - M, 0\}|\mathcal{F}_t)}_{=:V_3(t,k)}\underbrace{\mathrm{Var}\Big(\frac{S_k(S_k-l)}{u-l}|\mathcal{F}_t\Big)}_{=:V_4(t,k)}} - C_k\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\,\Big(E_3(t,k)\cdot E_4(t,k) + \rho_2 \cdot \sqrt{V_3(t,k)\cdot V_4(t,k)} - C_k\Big)\,\mathrm{d}k, \qquad (26)
\end{aligned}
$$

where $\rho_2 = \mathrm{corr}(\max\{P_k - M, 0\}, \frac{S_k(S_k-l)}{u-l})$.

From above we know the values E_3 and V_3. Let us compute the values E_4 and V_4. We start with

$$
\begin{aligned}
E_4(t,k) &:= \mathbb{E}\Big[\frac{S_k(S_k-l)}{u-l}|\mathcal{F}_t\Big] \\
&= \frac{1}{\Delta}\mathbb{E}\big[S_k^2|\mathcal{F}_t\big] - \frac{l}{\Delta}\mathbb{E}\big[S_k|\mathcal{F}_t\big] \\
&= \frac{1}{\Delta}\mathbb{E}\big[(S_k-m)^2|\mathcal{F}_t\big] + \frac{u}{\Delta}\mathbb{E}\big[S_k - m|\mathcal{F}_t\big] + \frac{m}{2} \\
&= \frac{1}{\Delta}\int_l^u (S_k-m)^2 p_{k-t}(S_t,S_k)\,\mathrm{d}S_k + \frac{u}{\Delta}E_0 + \frac{m}{2} \\
&= \frac{1}{\Delta}\int_l^u (y-m)^2 p_{k-t}(x,y)\,\mathrm{d}y + \frac{u}{\Delta}E_0 + \frac{m}{2} \\
&= \frac{1}{\Delta}\Big(\frac{\Delta^2}{12} + \big((x-m)^2 - \frac{\Delta^2}{12}\big)e^{-6(k-t)}\Big) + \frac{4}{\Delta}(x-m)e^{-2(k-t)} + \frac{m}{2} \\
&= \Big(\frac{(x-m)^2}{\Delta} - \frac{\Delta}{12}\Big)e^{-6(k-t)} + \frac{4}{\Delta}(x-m)e^{-2(k-t)} + \frac{2u-l}{6} \\
&= \Big(\frac{(S_t-m)^2}{\Delta} - \frac{\Delta}{12}\Big)e^{-6(k-t)} + \frac{4}{\Delta}(S_t-m)e^{-2(k-t)} + \frac{2u-l}{6}. \qquad (27)
\end{aligned}
$$

Now we continue with V_4

$$\begin{aligned} V_4(t,k) &:= \mathrm{Var}\Big(\frac{S_k(S_k-l)}{u-l}\big|\mathcal{F}_t\Big) \\ &= \frac{1}{\Delta^2} Var(S_k^2 - lS_k\big|\mathcal{F}_t) \\ &= \frac{1}{\Delta^2}\int_l^u (S_k^2 - lS_k)^2 p_{k-t}(S_t,S_k)\,\mathrm{d}S_k - E_4^2(t,k) \\ &= \frac{1}{\Delta^2}\int_l^u y^4\, p_{k-t}(x,y)\,\mathrm{d}y - \frac{2l}{\Delta^2}\int_l^u y^3\, p_{k-t}(x,y)\,\mathrm{d}y + \frac{l^2}{\Delta^2}\int_l^u y^2\, p_{k-t}(x,y)\,\mathrm{d}y - E_4^2(t,k). \end{aligned} \tag{28}$$

The first three terms mainly involve the following expression for some a

$$y(a,n) := \int_l^u y^a P_n\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{\Delta}{2}\int_{-1}^1 \Big(\frac{\Delta}{2}(z+1)+l\Big)^a P_n(z)\,\mathrm{d}z = 0, \tag{29}$$

when $a < n$. So we need to compute the following integrals to finish solving Equation (28)

$$y(2,0) := \int_l^u y^2 P_0\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{u^3-l^3}{3}, \tag{30}$$

$$y(2,1) := \int_l^u y^2 P_1\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^2}{6}(u+l), \tag{31}$$

$$y(2,2) := \int_l^u y^2 P_2\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^3}{30}, \tag{32}$$

$$y(3,0) := \int_l^u y^3 P_0\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{u^4-l^4}{4}, \tag{33}$$

$$y(3,1) := \int_l^u y^3 P_1\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^2}{20}(3u^2+4lu+3l^2), \tag{34}$$

$$y(3,2) := \int_l^u y^3 P_2\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^3}{20}(u+l), \tag{35}$$

$$y(3,3) := \int_l^u y^3 P_3\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^4}{140}, \tag{36}$$

$$y(4,0) := \int_l^u y^4 P_0\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{u^5-l^5}{5}, \tag{37}$$

$$y(4,1) := \int_l^u y^4 P_1\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^2}{15}(2u^3+3lu^2+3ul^2+2l^3), \tag{38}$$

$$y(4,2) := \int_l^u y^4 P_2\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^3}{35}(2u^2+3lu+2l^2), \tag{39}$$

$$y(4,3) := \int_l^u y^4 P_3\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^4}{70}(u+l), \tag{40}$$

$$y(4,4) := \int_l^u y^4 P_4\Big(\frac{2}{\Delta}(y-l)-1\Big)\,\mathrm{d}y = \frac{(u-l)^5}{630}. \tag{41}$$

297 Let us now continue with Equation (28)

$$\begin{aligned}
V_4 &:= \mathrm{Var}\Big(\frac{S_k(S_k-l)}{u-l}\Big|\mathcal{F}_t\Big)\\
&= \frac{1}{\Delta^2}\int_l^u y^4 p_{k-t}(x,y)\,\mathrm{d}y - \frac{2l}{\Delta^2}\int_l^u y^3 p_{k-t}(x,y)\,\mathrm{d}y + \frac{l^2}{\Delta^2}\int_l^u y^2 p_{k-t}(x,y)\,\mathrm{d}y - E_4^2(t,k)\\
&= C(4,k-t,x)\frac{1}{\Delta^2}y(4,4)\\
&+ C(3,k-t,x)\Big(\frac{1}{\Delta^2}y(4,3)-\frac{2l}{\Delta^2}y(3,3)\Big)\\
&+ C(2,k-t,x)\Big(\frac{1}{\Delta^2}y(4,2)-\frac{2l}{\Delta^2}y(3,2)+\frac{l^2}{\Delta^2}y(2,2)\Big)\\
&+ C(1,k-t,x)\Big(\frac{1}{\Delta^2}y(4,1)-\frac{2l}{\Delta^2}y(3,1)+\frac{l^2}{\Delta^2}y(2,1)\Big)\\
&+ C(0,k-t,x)\Big(\frac{1}{\Delta^2}y(4,0)-\frac{2l}{\Delta^2}y(3,0)+\frac{l^2}{\Delta^2}y(2,0)\Big) - E_4^2(t,k)\\
&= \frac{1}{\Delta^2}\times\begin{pmatrix} C(4,k-t,x)\\ C(3,k-t,x)\\ C(2,k-t,x)\\ C(1,k-t,x)\\ C(0,k-t,x)\end{pmatrix}^T \times \begin{pmatrix} y(4,4) & 0 & 0\\ y(4,3) & y(3,3) & 0\\ y(4,2) & y(3,2) & y(2,2)\\ y(4,1) & y(3,1) & y(2,1)\\ y(4,0) & y(3,0) & y(2,0)\end{pmatrix} \times \begin{pmatrix} 1\\ -2l\\ l^2\end{pmatrix} - E_4^2(t,k), \qquad (42)
\end{aligned}$$

298 where $C(n,x,k-t) := \frac{2n+1}{\Delta}P_n\Big(\frac{2}{\Delta}(x-l)-1\Big)e^{-n(n+1)(k-t)}$.

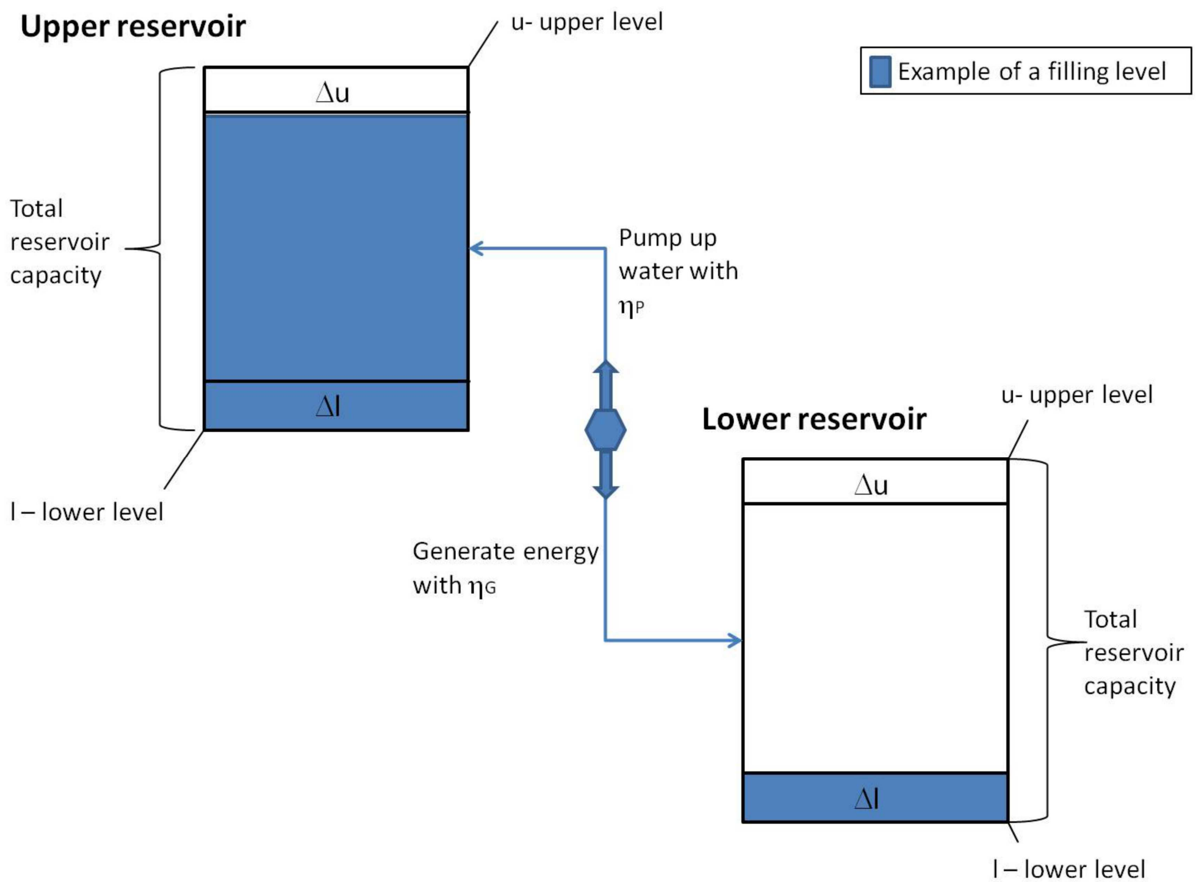


FIGURE 2. Pumped storage reservoir.

3.4.3. *Payoff 3.* Let us now consider the hydro power station with two reservoirs R_1 and R_2. The scheme of this pumped storage example is given in Figure 2. We make the following assumptions:

- the inflow to R_1 is random, since it depends on precipitation and thaw;
- no other water inflow into the reservoirs is possible;
- by injection water from R_1 to R_2 we produce power and by pumping water up from R_2 to R_1 we fill up R_1 for our future production purposes when needed and possible;
- it takes more energy to pump water up than to produce energy;
- both reservoirs have the same capacity of (l, u);
- our hydro-driven power plant contains water of one full reservoir capacity;

Our assumptions yield that $u - S_t^1 = S_t^2 - l$. Then for some levels $l < K_1 \leq m \leq K_2 < u$ in R_1 and R_2 we can construct the payoff

$$H_t(P_t, S_t^1, S_t^2, C_t) = \max\{P_t - M^+, 0\} \max\{S_t^1 - K_1, 0\} - \max\{M^- - P_t, 0\} \max\{S_t^2 - K_2, 0\} - C_t$$

$$
\begin{aligned}
&= \max\{P_t - M^+, 0\} \max\{S_t^1 - K_1, 0\} - \max\{M^- - P_t, 0\} \max\{K_1 - S_t^1, 0\} - C_t \\
(43) \qquad &= H_t(P_t, S_t^1, C_t)
\end{aligned}
$$

Since due to physical reasons we assume that pumping up needs more energy than producing power, then there exists the so-called ΔP such that $M^- = M - \Delta P$ and $M^+ = M + \Delta P$, which yields that when $P_t \in (M^-, M^+)$ it is not profitable to generate or to buy power. This ΔP can be computed via average price M and efficiency rates of pumping and generating. The details are given in Connolly, Lund, Finn, Mathiesen, and Leahy (2011).

This payoff has the following interpretation

- **Case 1:** $P_t > M^+$ **(power prices are relatively high) and** $S_t^1 > K_1$ **(reservoir** R_1 **is relatively full which implies that reservoir** R_2 **is relatively empty).** This is the most favorable situation which results in a positive value that investor has by releasing some water into reservoir R_2, producing power and selling it at the market.

- **Case 2:** $P_t > M^+$ **(power prices are relatively high) and** $S_t^1 < K_1$ **(reservoir** R_1 **is relatively empty which implies that reservoir** R_2 **is relatively full).** This situation is quite unfavorable, since to produce power we first need to pump the water up from R_2 to R_1. We need to buy power to do so, but since the prices are high we don't do anything.

- **Case 3:** $P_t < M^-$ **(power prices are relatively low) and** $S_t^2 > K_2$ **(reservoir** R_1 **is relatively empty which implies that reservoir** R_2 **is relatively full).** This is also a quite favorable situation for us, since we can buy power at a relatively low price and pump water immediately up to get the reservoir R_1 full.

- **Case 4:** $P_t < M^-$ **(power prices are relatively high) and** $S_t^2 < K_1$ **(reservoir** R_1 **is relatively full which implies that reservoir** R_2 **is relatively empty).** Here we are not interested to buy power since despite the price level our reservoir R_1 is already full and we don't need to pump it up.

Being at time t we find that

$$
\begin{aligned}
V_t(P_t, S_t^1, S_t^2, C_t) &= \mathbb{E}\Big[\int_t^T e^{-r(k,t)}\, H_k(P_k, S_k^1, S_t^2, C_k)\, \mathrm{d}k | \mathcal{F}_t\Big] \\
&= \int_t^T e^{-r(k,t)} \Big(\mathbb{E}\big[H_k(P_k, S_k^1, S_k^2, C_k) | \mathcal{F}_t\big]\Big)\, \mathrm{d}k \\
&= \int_t^T e^{-r(k,t)} \Big(\mathbb{E}\big[\max\{P_k - M^+, 0\} \times \max\{S_k^1 - K_1, 0\} \\
&- \max\{M^- - P_k, 0\} \times \max\{S_k^2 - K_2, 0\} | \mathcal{F}_t\big] - C_k\Big)\, \mathrm{d}k \\
&= \int_t^T e^{-r(k,t)} \Big(\mathbb{E}\big[\max\{P_k - M^+, 0\} \times \max\{S_k^1 - K_1, 0\} \\
&- \max\{M^- - P_k, 0\} \times \max\{K_1 - S_k^1, 0\} | \mathcal{F}_t\big] - C_k\Big)\, \mathrm{d}k
\end{aligned}
$$

$$
\begin{aligned}
&= \int_t^T e^{-r(k,t)} \Big(\underbrace{\mathbb{E}\big[\max\{P_k - M^+, 0\} | \mathcal{F}_t\big]}_{=:\tilde{E}_3(t,k)} \underbrace{\mathbb{E}\big[\max\{S_k^1 - K_1, 0\} | \mathcal{F}_t\big]}_{=:E_8(t,k)} \\
&- \underbrace{\mathbb{E}\big[\max\{M^- - P_k, 0\} | \mathcal{F}_t\big]}_{=:\tilde{E}_5(t,k)} \underbrace{\mathbb{E}\big[\max\{K_1 - S_k^1, 0\} | \mathcal{F}_t\big]}_{=:E_{10}(t,k)} \\
&+ \rho_3 \Big(\sqrt{\underbrace{\mathrm{Var}(\max\{P_k - M^+, 0\} | \mathcal{F}_t)}_{=:\tilde{V}_3(t,k)} \underbrace{\mathrm{Var}(\max\{S_k^1 - K_1, 0\} | \mathcal{F}_t)}_{=:V_8(t,k)}} \\
&- \sqrt{\underbrace{\mathrm{Var}(\max\{M^- - P_k, 0\} | \mathcal{F}_t)}_{=:\tilde{V}_5(t,k)} \underbrace{\mathrm{Var}(\max\{K_1 - S_k^1, 0\} | \mathcal{F}_t)\Big)}_{=:V_{10}(t,k)}} - C_k \Big) \,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)} \Big(\tilde{E}_3(t,k) \cdot E_8(t,k) - \tilde{E}_5(t,k) \cdot E_{10}(t,k) \\
&+ \rho_3 \Big(\sqrt{\tilde{V}_3(t,k) \cdot V_8(t,k)} - \sqrt{\tilde{V}_5(t,k) \cdot V_{10}(t,k)} \Big) - C_k \Big), \,\mathrm{d}k. \qquad (44)
\end{aligned}
$$

where we assume $\rho_3 = \mathrm{corr}(\max\{P_k - M^+, 0\}, \max\{S_k^1 - K_1, 0\}) = \mathrm{corr}(\max\{M^- - P_k, 0\}, \max\{S_k^2 - K_2, 0\})$.

The expressions for $\tilde{E}_3$ and $\tilde{V}_3$ can be obtained from E_3 and V_3 given above by replacing M by M^+. The same way the expressions for $\tilde{E}_5$ and $\tilde{V}_5$ can be obtained from E_5 and V_5 given below by replacing M by M^-. Now let us compute E_8, E_{10}, V_8 and V_{10}. We start with E_8 which is analogous to (19)

$$
\begin{aligned}
E_8(t,k) &:= \mathbb{E}\big[\max\{S_k^1 - K_1, 0\} | \mathcal{F}_t\big] \\
&= \int_{K_1}^u (y - K_1) p_{t-k}(y,x) \,\mathrm{d}y \\
&= \frac{1}{\Delta} \Big(\frac{\Delta^2}{4} \frac{1 - \tilde{K}_1^2}{2} + \frac{\Delta}{2} (m - K_1)(1 - \tilde{K}) \Big) \\
&+ \frac{3}{\Delta} P_1 \big(\frac{2}{\Delta} (S_t^1 - l) - 1 \big) e^{-2(T-t)} \Big(\frac{\Delta^2}{4} \frac{1 - \tilde{K}_1^3}{3} + \frac{\Delta}{2} (m - K_1) \frac{1 - \tilde{K}_1^2}{2} \Big) \\
&+ \sum_{n=2}^{\infty} \frac{2n+1}{\Delta} P_n \left(\frac{2}{\Delta} (S_t^1 - l) - 1 \right) e^{-n(n+1)(T-t)} \Big(\frac{\Delta^2}{4} P^*_{n-2,n,n+2}(\tilde{K}_1) + \frac{\Delta}{2} (m - K_1) P^*_{n-1,n+1}(\tilde{K}) \Big) \qquad (45)
\end{aligned}
$$

where $\tilde{K}_1 := \frac{2}{\Delta}(K_1 - l) - 1$ and expressions for $P^*_{n-2,n,n+2}(x)$ and $P^*_{n-1,n+1}(x)$ are given above in Equations (9) and (8) respectively. Then

$$
\begin{aligned}
E_{10}(t,k) &:= \mathbb{E}\big[\max\{S_k^2 - K_2, 0\} | \mathcal{F}_t\big] \\
&= \mathbb{E}\big[\max\{K_1 - S_k^1, 0\} | \mathcal{F}_t\big] \\
&= K_1 + m - E_0(t,k) - E_8(t,k). \qquad (46)
\end{aligned}
$$

We continue with V_8 and V_{10}. When $K_1 = m$ the case is identical to $V_2(t,k)$, but here we assume that K_1 is different from m, then

$$\begin{aligned} V_8(t,k) &:= \mathrm{Var}\Big(\max\{S_k^1 - K_1, 0\}|\mathcal{F}_t\Big) \\ &= \int_{K_1}^{u} (y-K_1)^2 p_{t-k}(x,y)\ \mathrm{d}y - E_8^2(t,k) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{\Delta} P_n\left(\frac{2}{\Delta}(x-l)-1\right) e^{-n(n+1)(k-t)} \int_{K_1}^{u} (y-K_1)^2 P_n\left(\frac{2}{\Delta}(y-l)-1\right)\ \mathrm{d}y - E_8^2(t,k), \end{aligned} \tag{47}$$

now substituting $z := \frac{2}{\Delta}(y-l)-1$ and $\tilde{K}_1 := \frac{2}{\Delta}(K_1-l)-1$ yields

$$\begin{aligned} \int_{K_1}^{u} (y-K_1)^2 P_n\left(\frac{2}{\Delta}(y-l)-1\right)\ \mathrm{d}y &= \frac{\Delta}{2}\int_{\tilde{K}_1}^{1}\left(\frac{\Delta}{2}z + (m-K_1)\right)^2 P_n(z)\ \mathrm{d}z \\ &= \frac{\Delta^3}{8}\int_{\tilde{K}_1}^{1} z^2 P_n(z)\ \mathrm{d}z + \frac{\Delta^2}{2}(m-K_1)\int_{\tilde{K}_1}^{1} zP_n(z)\ \mathrm{d}z \\ &+ \frac{\Delta}{2}(m-K_1)^2\int_{\tilde{K}_1}^{1} P_n(z)\ \mathrm{d}z \\ &= \frac{\Delta^3}{8}P^*_{n-3,n+3}(\tilde{K}_1) + \frac{\Delta^2}{2}(m-K_1)P^*_{n-2,n,n+2}(\tilde{K}_1) \\ &+ \frac{\Delta}{2}(m-K_1)^2 P^*_{n-1,n+1}(\tilde{K}_1) \end{aligned} \tag{48}$$

where we used Equations (8), (9) and (10). Now we continue with solving Equation (47)

$$\begin{aligned} V_8(t,k) &= \frac{1}{\Delta}\Big(\frac{\Delta^3}{8}\frac{1-\tilde{K}_1^3}{3} + \frac{\Delta^2}{2}(m-K_1)\frac{1-\tilde{K}_1^2}{2} + \frac{\Delta}{2}(m-K_1)^2(1-\tilde{K}_1)\Big) \\ &+ \frac{3}{\Delta}P_1\left(\frac{2}{\Delta}(x-l)-1\right)e^{-2(k-t)}\Big(\frac{\Delta^3}{8}\frac{1-\tilde{K}_1^4}{4} + \frac{\Delta^2}{2}(m-K_1)\frac{1-\tilde{K}_1^3}{3} + \frac{\Delta}{2}(m-K_1)^2\frac{1-\tilde{K}_1^2}{2}\Big) \\ &+ \frac{5}{\Delta}P_2\left(\frac{2}{\Delta}(x-l)-1\right)e^{-6(k-t)}\Big(\frac{\Delta^3}{8}\frac{-9\tilde{K}_1^5+5\tilde{K}_1^3+4}{30} + \frac{\Delta^2}{2}(m-K_1)\frac{-3\tilde{K}_1^4+2\tilde{K}_1^2+1}{8} \\ &+ \frac{\Delta}{2}(m-K_1)^2\frac{\tilde{K}_1-\tilde{K}_1^3}{2}\Big) \\ &+ \sum_{n=3}^{\infty}\frac{2n+1}{\Delta}P_n\left(\frac{2}{\Delta}(x-l)-1\right)e^{-n(n+1)(k-t)}\Big(\frac{\Delta^3}{8}P^*_{n-3,n+3}(\tilde{K}_1) + \frac{\Delta^2}{2}(m-K_1)P^*_{n-2,n,n+2}(\tilde{K}_1) \\ &+ \frac{\Delta}{2}(m-K_1)^2 P^*_{n-1,n+1}(\tilde{K}_1)\Big) - E_8^2(t,k). \end{aligned} \tag{49}$$

Now the last element is V_{10}

$$\begin{aligned}
V_{10}(t,k) &:= \mathrm{Var}\big(\max\{K_1 - S_k^1, 0\}|\mathcal{F}_t\big) \\
&= \int_l^{K_1} (K_1 - y)^2 p_{t-k}(x,y)\ \mathrm{d}y - E_{10}^2(t,k) \\
&= \int_l^{u} (K_1 - y)^2 p_{t-k}(x,y)\ \mathrm{d}y - \int_{K_1}^{u} (K_1 - y)^2 p_{t-k}(x,y)\ \mathrm{d}y - E_{10}^2(t,k) \\
&= V_0(k,t) + \big(K_1 + m - E_0(k,t)\big)^2 - (V_8(t,k) + E_8^2(t,k)) - E_{10}^2(t,k). \qquad (50)
\end{aligned}$$

3.5. **Gas-Driven Storage.** In this section we consider one gas-storage-driven payoff which is similar to the payoffs studied above. Consider a storage owner who regularly sells or buys gas and respectively fills or empties the storage facility. Assume, that this is her stochastic optimal control policy and the decision to inject or withdraw is a result of the optimisation problem under some constraints. The power price and costs on the managing storage facility are the key drivers to find the optimal policy. If taking the costs as a deterministic function of time, one can think of this policy as solely dependent of the stochastic gas price. So we can further assume that the resulting storage level S_t is indirectly a function of the optimal stochastic control. In this sense we can consider a payoff that would be hedging the position of this storage owner in case of a low reservoir level and low prices.

3.5.1. *Payoff 4.* In this section we investigate the case of the gas storage problem. We consider the so-called working gas facility and the power producer who can inject and withdraw the necessary amount of gas into the reservoir. We also have a deterministic cost function C_t

$$H_t(P_t, S_t, C_t) = \max\{P_t - M, 0\} \times \max\{S_t - m, 0\} - \max\{M - P_t, 0\} \times \max\{m - S_t, 0\} - C_t, \qquad (51)$$

where M is again the average gas price level and all the notations are as given above. This payoff has the following interpretation:

- **Case 1:** $P_t > M$ **(gas prices are relatively high) and** $S_t > m$ **(reservoir is full).** This is the most favorable situation which results in a positive value that investor can have by withdrawing and selling the storable asset at the market.
- **Case 2:** $P_t > M$ **(gas prices are relatively high) and** $S_t < m$ **(reservoir is relatively empty).** This situation corresponds to "doing nothing" regime, since the prices are too high to buy and inject and the reservoir is too low to withdraw and sell. So, the value is negative due to the costs we have to pay.
- **Case 3:** $P_t < M$ **(gas prices are relatively low) and** $S_t > m$ **(reservoir is full).** This situation also corresponds to "doing nothing" regime, since prices are too low to withdraw and sell despite the fact that the reservoir is full. So, the value is negative due to the costs we have to pay.

- **Case 4: $P_t < M$ (gas prices are relatively low) and $S_t < m$ (reservoir is relatively empty).** This is an auspicious situation for an investor to buy and inject the storable asset, although the value is negative.

Being at time t we find that

$$
\begin{aligned}
V_t(P_t,S_t,C_t) &= \mathbb{E}\Big[\int_t^T e^{-r(k,t)}\,H_k(P_k,S_k,C_k)\,\mathrm{d}k|\mathcal{F}_t\Big] \\
&= \int_t^T e^{-r(k,t)}\Big(\mathbb{E}\big[H_k(P_k,S_k,C_k)|\mathcal{F}_t\big]\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\Big(\mathbb{E}\big[\max\{P_k-M,0\}\times\max\{S_k-m,0\} \\
&- \max\{M-P_k,0\}\times\max\{m-S_k,0\}-C_k|\mathcal{F}_t\big]\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\Big(\underbrace{\mathbb{E}\big[\max\{P_k-M,0\}|\mathcal{F}_t\big]}_{=:E_3(t,k)}\,\underbrace{\mathbb{E}\big[\max\{S_k-m,0\}|\mathcal{F}_t\big]}_{=:E_2(t,k)} \\
&- \underbrace{\mathbb{E}\big[\max\{M-P_k,0\}|\mathcal{F}_t\big]}_{=:E_5(t,k)}\,\underbrace{\mathbb{E}\big[\max\{m-S_k,0\}|\mathcal{F}_t\big]}_{=:E_6(t,k)} \\
&+ \rho_4\Big(\sqrt{\underbrace{\mathrm{Var}(\max\{P_k-M,0\}|\mathcal{F}_t)}_{=:V_3(t,k)}\,\underbrace{\mathrm{Var}(\max\{S_k-m,0\}|\mathcal{F}_t)}_{=:V_2(t,k)}} \\
&- \sqrt{\underbrace{\mathrm{Var}(\max\{M-P_k,0\}|\mathcal{F}_t)}_{=:V_5(t,k)}\,\underbrace{\mathrm{Var}(\max\{m-S_k,0\}|\mathcal{F}_t)\Big)}_{=:V_6(t,k)}}-C_k\Big)\,\mathrm{d}k \\
&= \int_t^T e^{-r(k,t)}\Big(E_3(t,k)\cdot E_2(t,k)-E_5(t,k)\cdot E_6(t,k) \\
&+ \rho_4\Big(\sqrt{V_3(t,k)\cdot V_2(t,k)}-\sqrt{V_5(t,k)\cdot V_6(t,k)}\Big)-C_k\Big)\,\mathrm{d}k,
\end{aligned}
\tag{52}
$$

where we assume $\rho_4 = \mathrm{corr}(\max\{P_k-M,0\},\max\{S_k-m,0\}) = \mathrm{corr}(\max\{M-P_k,0\},\max\{S_k-m,0\})$.

We need to compute values E_5, E_6, V_5 and V_6. But this can be done easily since we know the values E_2, E_3, V_2, and V_3 from the previous sections. So we start with E_5

$$
\begin{aligned}
E_5(t,k) &:= \mathbb{E}\big[\max\{M-P_k,0\}|\mathcal{F}_t\big] \\
&= \mathbb{E}\big[\max\{M-e^{f(k)+X_k},0\}|\mathcal{F}_t\big] \\
&= \int_{-\infty}^{\ln M}(M-e^y)\phi(y,x,t)\;\mathrm{d}y \\
&= \int_{-\infty}^{+\infty}(M-e^y)\phi(y,x,t)\;\mathrm{d}y+E_3
\end{aligned}
$$

$$
\begin{aligned}
&= M - e^{m_1+\frac{m_2}{2}} + e^{m_1+\frac{m_2}{2}}\Phi(d_2) - M\Phi(d_1) \\
&= M\Phi(-d_1) + e^{m_1+\frac{m_2}{2}}\Phi(-d_2), \qquad (53)
\end{aligned}
$$

where d_1 and d_2 are given above and continue with V_5 recalling that $y := f(k) + X_k \sim \mathcal{N}(m_1, m_2)$ with m_1, m_2 and d_3 given above

$$
\begin{aligned}
V_5(t,k) &:= \mathrm{Var}(\max\{M - P_k, 0\}|\mathcal{F}_t) \\
&= \mathrm{Var}(\max\{M - e^{f(k)+X_k}, 0\}|\mathcal{F}_t) \\
&= \int_{-\infty}^{\ln M} (M-e^y)^2\phi(y,x,t)\ \mathrm{d}y - \Big(\int_{-\infty}^{\ln M} (M-e^y)\phi(y,x,t)\ \mathrm{d}y\Big)^2 \\
&= \int_{-\infty}^{+\infty} (M-e^y)^2\phi(y,x,t)\ \mathrm{d}y - \int_{\ln M}^{+\infty} (M-e^y)^2\phi(y,x,t)\ \mathrm{d}y - E_5^2(t,k) \\
&= \mathrm{Var}(M-e^y) + \mathbb{E}^2[M-e^y] - \int_{\ln M}^{+\infty} (e^y-M)^2\phi(y,x,t)\ \mathrm{d}y - E_5^2(t,k) \\
&= \mathrm{Var}(e^y) + (M-\mathbb{E}[e^y])^2 - \int_{\ln M}^{+\infty} (e^y-M)^2\phi(y,x,t)\ \mathrm{d}y - E_5^2(t,k) \\
&= (e^{m_2}-1)e^{2m_1+m_2} + \big(M - e^{m_1+\frac{m_2}{2}}\big)^2 - (V3 + E_3^2) - E_5^2(t,k). \qquad (54)
\end{aligned}
$$

Then we continue with E_6

$$
\begin{aligned}
E_6(t,k) &:= \mathbb{E}\big[\max\{m - S_k, 0\}|\mathcal{F}_t\big] \\
&= \int_l^m (m-y)p_{k-t}(x,y)\ \mathrm{d}y \\
&= E_2(t,k) - E_0(t,k). \qquad (55)
\end{aligned}
$$

We finally proceed with V_6

$$
\begin{aligned}
V_6(t,k) &:= \mathrm{Var}(\max\{m - S_k, 0\}|\mathcal{F}_t) \\
&= \int_l^m (m-S_k)^2 p(S_k,k,S_t,t)\,\mathrm{d}S_k - E_6^2(t,k) \\
&= \int_l^m (m-y)^2 p_{k-t}(x,y)\,\mathrm{d}y - E_6^2(t,k) \\
&= \int_l^u (m-y)^2 p_{k-t}(x,y)\,\mathrm{d}y - \int_m^u (m-y)^2 p_{k-t}(x,y)\,\mathrm{d}y - E_6^2(t,k) \\
&= V_0(t,k) - V_2(t,k) + E_0^2(t,k) - E_2^2(t,k) - E_6^2(t,k). \qquad (56)
\end{aligned}
$$

4. Numerical Examples

4.1. **Technical points.** Since all the payoffs include elements with infinite sums, we need to clarify the choice for the number of terms in the sum. There are two issues here: convergence and computational time. The detailed discussion is given in **?**.

4.2. **Hydro storage.**

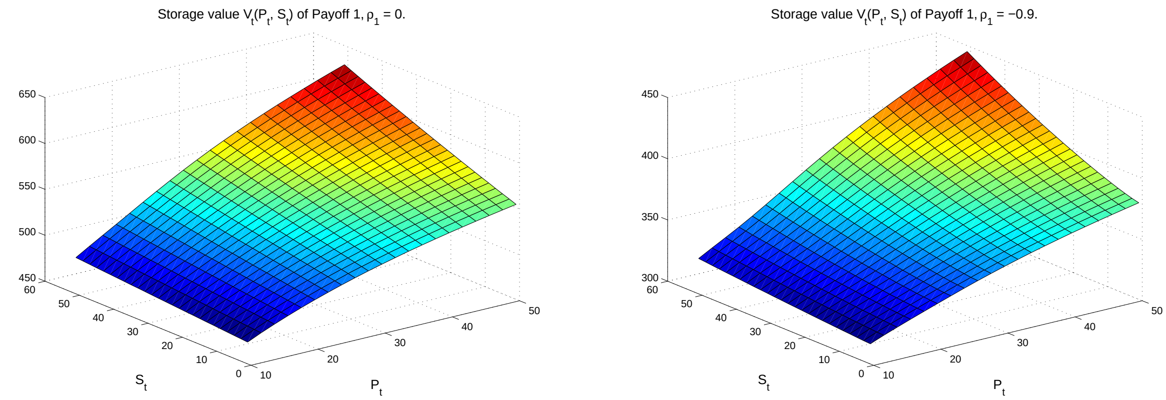
4.2.1. *Simple products.* In section 3.3 we discussed fundamental financial products that can be used for hedging purposes in the storage industry. One can easily check that the formulas for the futures and options demand negligible computational effort.

4.2.2. *Hydro-Driven Storage: Payoffs 1, 2, 3.* In this section we illustrate the hydro storage value problem described above by various payoffs. For some fixed parameters values we plot the payoff for a range of S_t and P_t. For all the examples we consider $t = 0.5$ and $T = 5$ in years. For the sake of simplicity we also fix the discount factor $r(t,T)$ and the costs of storage maintenance C_t. We take the following parameters values: $l = 1$, $u = 51$, $m = 26$, $\alpha = 1.5$, $\sigma = 0.2$, $M = 30$, $C = 0$ and $r = 0.03$.

To investigate the role of correlation parameters ρ_1, ρ_2 and ρ_3, we study two cases: zero and negative correlation. Zero correlation implies that there is no any relationship between the power price level and the storage level. More precisely, when the inflow increases, the power price neither decreases nor increases. In the markets where different fossil fuels (coal, gas) dominate over hydro-driven power production, zero or negligible correlation can be exactly the case since there are many other power price drivers apart from the current reservoir level. However, in the markets with significant or even dominating share of hydro facilities we can fairly expect negative correlation. When the inflow increases and the so-called cumulative reservoir is getting full of water, the supply uncertainty decreases and all the market participants are aware of this. So since there is no lack of water in the reservoir, the power price decreases. We will find what is the impact of the correlation on the price of our financial instruments. So, this can be considered as a correlation sensitivity analysis.

Figure 3 depicts the value associated with the payoff 1 as given in Equation (21) for various values of the correlation parameter ρ_1. We observe that relatively high power prices and a full water reservoir yield the highest possible profit. We also observe that relatively low power price and an empty water reservoir yield the lowest possible profit. We further note that with negative correlation $\rho_1 = -0.9$ producer's profit decreases compared with $\rho_1 = 0$. We interpret this gap as a premium that a producer has to pay for the market information about current reservoir level.

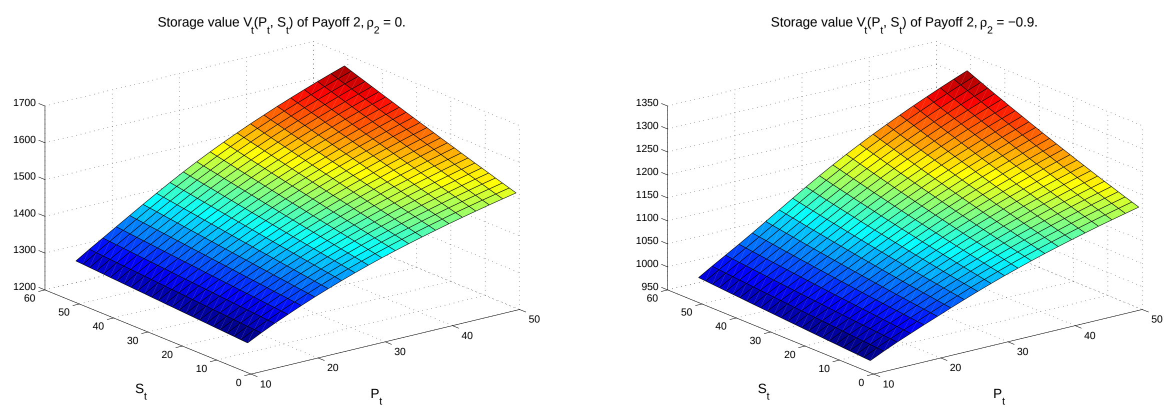
Figure 4 depicts the value associated with the payoff 2 as given in Equation (26) for various values of the correlation parameter ρ_2. We again observe that relatively high power prices and a full water reservoir yield the highest possible profit and that relatively low power price and an empty water reservoir yield the lowest possible profit. We also see the same negative correlation effect. The main difference here is that the profit of the payoff 2 is almost three times higher compared to the profit of the payoff 1. We reason this with the



(A) (B)

FIGURE 3. Storage value with payoff 1 with power price P_t and storage level S_t. Parameters: $l = 1$, $u = 51$, $m = 26$, $\alpha = 1.5$, $\sigma = 0.2$, $M = 30$, $C = 0$, $r = 0.03$.

flexibility that offers the payoff 2: in a contrast to the payoff 1, the payoff 2 allows us to produce power even if the current reservoir level is less than m at some rate $\kappa(S_t)$.



(A) (B)

FIGURE 4. Storage value with payoff 2 with power price P_t and storage level S_t. Parameters: $l = 1$, $u = 51$, $m = 26$, $\alpha = 1.5$, $\sigma = 0.2$, $M = 30$, $C = 0$, $r = 0.03$.

Figure 5 depicts the value associated with the payoff 3 as given in Equation (44) for various values of the correlation parameter ρ_3 and parameter K_1. We mainly observe two dependencies: the profit decrease when the correlation coefficient ρ_3 and/or the coefficient K_1 increases. The reasoning for the first case is the same as above: this can be regarded as an information premium for the producer. The explanation for the second case is intuitively clear: when the critical production level K_1 at which we are allowed to produce is low, we have larger capacity to produce and benefit. When the critical production value K_1 is high, we have much smaller capacity for power production. The value can even be negative and our producer has losses.

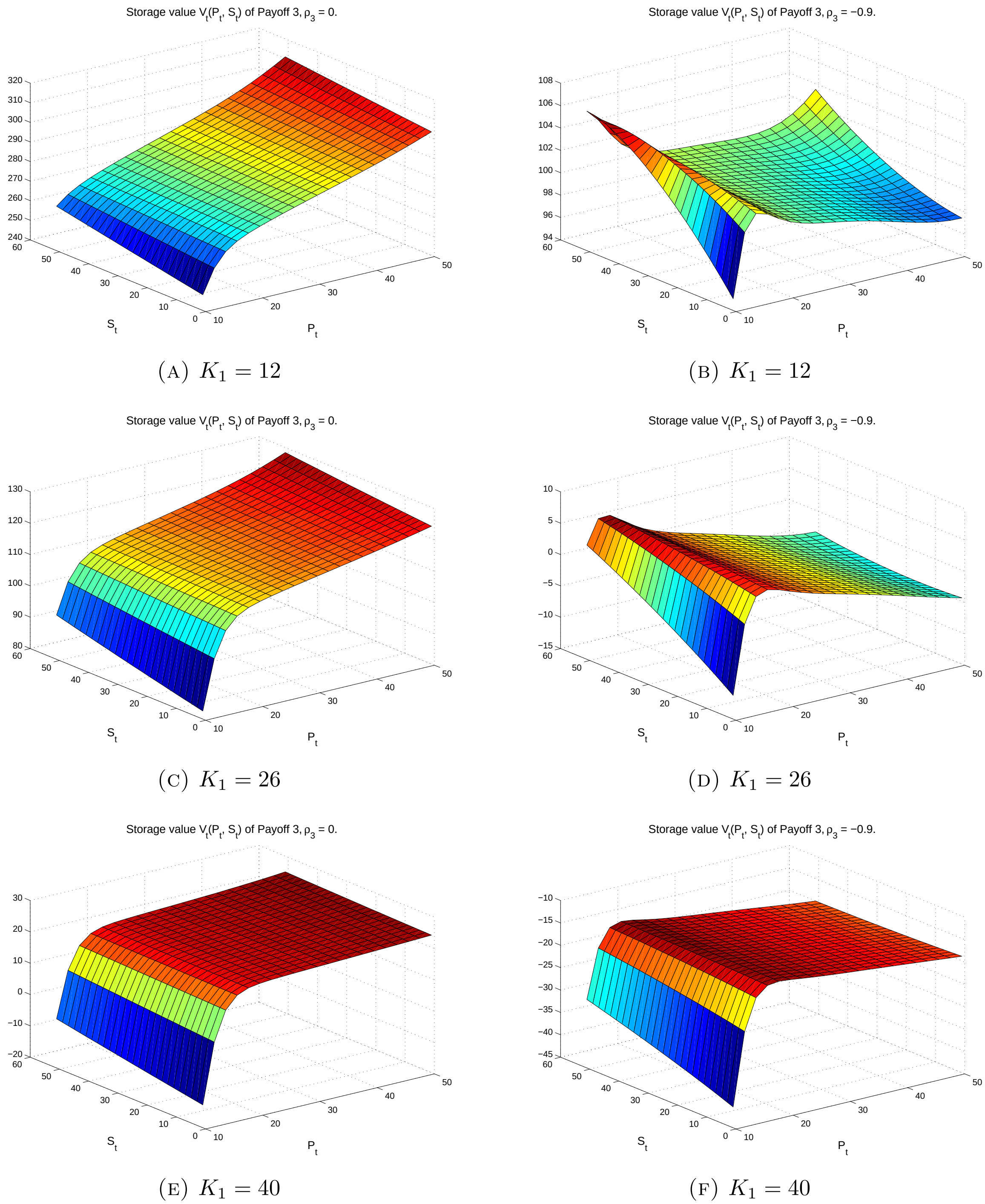


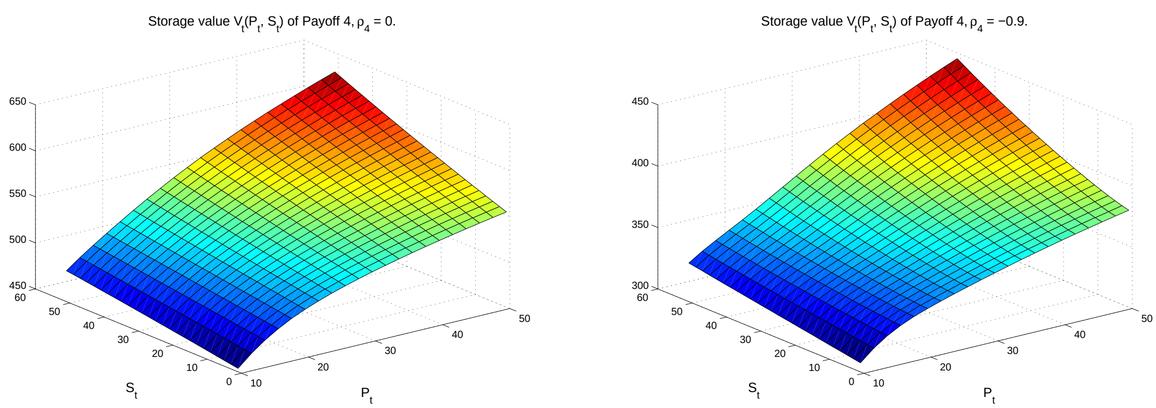
FIGURE 5. Storage value with payoff 3 with power price P_t and storage level S_t. Parameters: $l = 1$, $u = 51$, $m = 26$, $\alpha = 1.5$, $\sigma = 0.2$, $M^- = 25$, $M^+ = 50$, $C = 0$, $r = 0.03$.

4.3. **Gas storage.**

4.3.1. *Gas-Driven Storage: Payoff 4.* In this section we illustrate the gas storage value problem described by the payoff 4. For some fixed parameters values we plot the payoff for a range of S_t and P_t. For all the examples we consider $t = 0.5$ and $T = 5$ in years. For the sake of simplicity we also fix the discount factor $r(t, T)$ and the costs of storage maintenance C_t. We take the following parameters values: $l = 1$, $u = 51$, $m = 26$, $\alpha = 1.5$, $\sigma = 0.2$, $M = 30$, $C = 0$ and $r = 0.03$.

Here we also study two cases: zero and negative correlation coefficient ρ_4. However, here we assume that in the markets where different fossil fuels (coal, gas) dominate over hydro-driven power production, negative correlation can be exactly the case, since the market is aware of the current supply level. In the markets with dominating share of hydro facilities we can fairly expect zero correlation, since the current gas storage level will not be critical for power production.

Figure 6 depicts the value associated with the payoff 4 as given in Equation (52) for various values of the correlation parameter ρ_4. This result is consistent with the hydro-storage case considered in the previous section. Particularly, when the gas price is high and the current storage level is also high, the storage owner withdraws, sells the gas in the market and obtains profit. If the gas price is relatively low and the storage facility is relatively empty, then the profit becomes smaller. We also notice the same effect with the correlation coefficient ρ_4: the higher the correlation value, the smaller is our value.



(A) (B)

FIGURE 6. Storage value with payoff 4 with gas price P_t and storage level S_t. Parameters: $l = 1$, $u = 51$, $m = 26$, $\alpha = 1.5$, $\sigma = 0.2$, $M = 30$, $C = 0$, $r = 0.03$.

5. Discussion and outlook

5.0.2. *Hydro-Storage with time-dependent trend component and jump-diffusion process for power prices.* Assumption about the power price being the mean-reverting diffusion process is quite simplifying and far away from realistic. So the natural steps towards reasonable storage and power price modeling would be to include a time-dependent trend component $m(t)$ and a jump component for the power price. Namely,

$$\mathrm{d}S_t = -2(S_t - m(t))\ \mathrm{d}t + \sqrt{2(S_t - l)(u - S_t)}\ \mathrm{d}W_t^S, \tag{57}$$

where $m(t)$ could be some trigonometric function capturing seasonal behavior of the storage level. And

$$\begin{aligned} P_t &= e^{f(t)+X_t}, \\ \mathrm{d}X_t &= -\alpha X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t^X, \\ \mathrm{d}P_t &= \alpha\Big(\mu(t) - \log P_t\Big) P_t \,\mathrm{d}t + \sigma P_t \,\mathrm{d}W_t^X + J_t \,\mathrm{d}N_t, \end{aligned} \tag{58}$$

where N_t is a compound Poisson process with some finite intensity λ. We can also consider various jump size distributions: Gaussian, exponential, Pareto, Laplace (for the motivation for this choice please see the details in Benth, Kiesel, and Nazarova (2012) and Bannör, Kiesel, Nazarova, and Scherer (2012)). Depending on this choice we either can directly compute the payoff value or we need to simulate the processes P_t and S_t to get the value of the virtual hydro-driven power plant (VHDPP) in the same manner as in Bannör, Kiesel, Nazarova, and Scherer (2012). Namely,

$$VHDPP(t,T) = \frac{1}{N}\sum_{i=1}^{N} VHDPP_i(t,T), \tag{59}$$

where

$$VHDPP_i(t,T) = \int_t^T e^{-r(s-t)}\ \mathrm{payoff}_i(s)\ \mathrm{d}s. \tag{60}$$

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6. Conclusion

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